The World of Manipulatives

Archimedes Maths Hub (Research and Innovation Work Group NCP22-H)

The object of this document is to identify key aspects of how and why manipulatives are deployed in schools. Research has been included to support and validate the work by schools in England.

*"Many teachers that I have worked with over the course of the last year have been confused about what manipulatives to use in specific contexts and I have seen quite a bit of practice where one specific manipulative might appear to teach a given concept and then disappear never to be seen again in the classroom.***"**

"Manipulatives do have a place as computational tools to support various calculation strategies and demonstrational tools to expound a procedure but it is only when learners themselves use the artefacts to support their own sense making processes that they will begin to see their power as tools for calculation."

Manipulatives in the Primary Classroom *Jenni Back (2013)*

This quote featured in many conversations with school leaders. Inconsistency seemed to be prevalent in most school curriculums. Often where manipulatives were recommended in programmes of study, there seemed to be a lack of coherence with regards to consistency or fidelity, what may have been recommended in addition may not be in multiplication. The notion that manipulatives are 'computational tools' may place a different emphasis on how they are incorporated into the curriculum, pupils understanding and staffs awareness of the need for training.

Schools stated that the demands of the curriculum created challenges within classrooms, challenges that left teacher's feeling confused regarding the use of manipulatives. A school survey conducted by the Maths Hub revealed that teachers were looking for support regarding:

- Ideas in how to structure lessons so that mathematics effectively incorporated manipulatives.
- When would manipulatives be incorporated into lessons.
- Why manipulatives are used in lessons.
- How can they be effectively applied throughout school.

Ofsted's 2012 report 'Made to Measure'

Ofsted suggests that although manipulatives are used in some primary schools to support teaching and learning they are not used as effectively or as widely as they might be.

- Staff were using manipulatives 'widely but not necessarily 'effectively.

Education Endowment Foundation (EEF)

It is important that children have opportunities to engage in both free and structured play with manipulatives. However, practitioners must help children to understand the links between the manipulatives or representations and the mathematical ideas they represent.

As children's understanding of mathematical ideas develops, practitioners should encourage children to use pictures, symbols and more abstract diagrams to represent and communicate ideas and concepts.

Manipulatives (physical objects used to teach maths) and representations (such as number lines and graphs) can help pupils engage with mathematical ideas. However, manipulatives and representations are just tools: how they are used is essential. They need to be used purposefully and appropriately to have an impact.

There must be a clear rationale for using a particular manipulative or representation to teach a specific mathematical concept. Manipulatives should be temporary; they should act as a 'scaffold' that can be removed once independence is achieved.

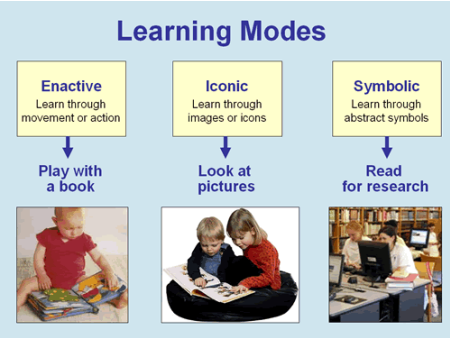
The EEF recommend that schools:

- Ensure that there is a clear rationale.
- Enable pupils to understand the links between the manipulatives and the mathematical ideas they represent.
- Try to avoid pupils becoming reliant on manipulatives to do a type of task or question.
- Manipulatives should act as a 'scaffold', which can be removed once independence is achieved.
- Manipulatives can be used to support pupils of all ages.

Jerome Bruner (1966)

The theory is based on the belief that all children, when introduced to a new concept, should have the opportunity to build competency by taking an Enactive, Iconic and Symbolic to learning. Children being provided with the opportunity to use concrete objects and resources to help them understand what they are doing, alongside use pictorial representations. These representations can then be used to help reason and solve problems to support children's understanding of abstract methods.

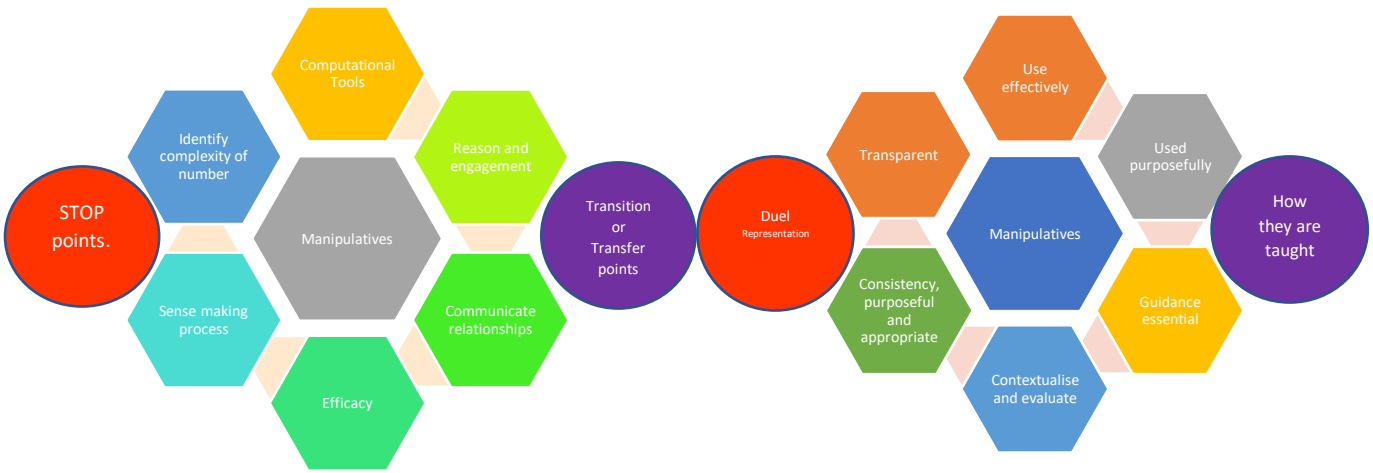
In 1966 as a means of scaffolding learning stated that the abstract nature of learning (and this is especially true in mathematics) is a "mystery" to many children. It, therefore, needs to be scaffolded by the use of effective representations.



In a Nutshell

It was agreed by the participants that manipulatives were a vital tool in supporting pupils to succeed in mathematics, providing opportunities to physically process how mathematics works and how concepts are grounded in the 'concrete' but are often applied in the abstract is a fundamental non-negotiable for all learners. Mathematics is a complex discipline and every opportunity to simply or set that in the 'real world' is vital.

The cycles identify key words that signify how important and how mindful, if used incorrectly (lacking in skill, expertise and understanding) manipulatives can influence a curriculum.



Research

Architecture of Mathematical Structure (Venkat, Askew, Watson, Mason 2019)

There is a notion that materials provide "representational affordances [that] emphasise either number notation conventions (such as bead strings, abacus and base-ten blocks) or number relations (such as Cuisenaire and Numicon)".

In contrast, cubes and counters are described as unstructured materials, as no mathematical relation inheres in their design, though they may become structured in use through formatting actions.

It is also worth noting the caveat that any 'in-building' of structure into resources does not guarantee that the relevant structure will be noticed or appropriated by the materials user.

Carbonneau, Marley and Selig (2013)

Findings indicate that using manipulatives in mathematics instruction produces a small- to medium-sized effect on student learning when compared with instruction that uses abstract symbols alone.

The strength of this effect is dependent upon other instructional variables such as

- the perceptual richness of an object,
- level of guidance offered to students during the learning process,
- the development status of the learner moderate the efficacy of manipulatives.

How Learning Theory Supports the Use of Manipulatives

The theory of experiential education revolves around the idea that learning is enhanced when pupils acquire knowledge through active processes that engage them (Hartshorn and Boren, 1990). Manipulatives can be key in providing effective, active, engaging lessons in the teaching of mathematics.

The use of manipulatives helps pupils hone their mathematical thinking skills. According to Stein and Bovalino (2001), "Manipulatives can be important tools in helping pupils to think and reason in more meaningful ways. By giving pupils concrete ways to compare and operate on quantities, such manipulatives as pattern blocks, tiles, and cubes can contribute to the development of well-grounded, interconnected understandings of mathematical ideas.

Understanding is necessary to evaluate new information; the more a person can contextualise what they're being told, and evaluate it from many angles, the less likely they are misled by manipulative language, ill-informed data, poorly sourced evidence or pure advertising. Ephrat Livni 2017

Identifying the complexity arising from the representation of numbers and relationships through the use of manipulatives is a cause for concern and recommends that teacher promotion of this important component of a focus on noticing must include support to enable pupils to use manipulatives to construct and communicate the significant aspects of the relationships or issue will continue. (Valid and Valuable) Barclay 2021

Mathematics achievement increases when manipulatives are put to good use. Clements and Battista, 1990; Skemp, 1987

How Research from the Classroom Supports the Use of Manipulatives

Over the past four decades, studies have been conducted at all different grade levels and in several different countries, they indicate that mathematics achievement increases when manipulatives are put to good use. (Clements and Battista, 1990; Skemp, 1987).

Manipulatives have also been shown to provide a strong foundation for pupils mastering the following mathematical concepts (The Access Center, October 1, 2004):

- Number Relations/Bases

- Measurement
- Decimals/Percentages
- Probability
- Statistics

How Research from the Classroom Supports the Use of Manipulatives

With long-term use of manipulatives in mathematics, educators have found that pupils make gains in the following general areas (Heddens; Picciotto, 1998; Sebesta and Martin, 2004):

- verbalising mathematical thinking
- discussing mathematical ideas and concepts
- relating real-world situations to mathematical symbolism
- working collaboratively
- thinking divergently to find a variety of ways to solve problems
- expressing problems and solutions using a variety of mathematical symbols
- making presentations
- taking ownership of their learning experiences
- gaining confidence in their abilities to find solutions to mathematical problems using methods that they come up with themselves without relying on directions from the teacher

What Makes Mathematics Manipulatives Effective? Lessons From Cognitive Science and Montessori Education (Sage)

Despite the widespread use of manipulatives in early childhood mathematics instruction, research examining the efficacy of manipulatives for mathematics instruction is inconsistent.

In fact, a recent large meta-analysis of studies that compared instruction with or without manipulatives indicated that instruction with manipulatives was least effective for children between the ages of 3 and 6 years, with very small and sometimes negative effects (Carbonneau et al., 2013).

Thus, it is imperative that early childhood educators think carefully about ways to effectively use mathematics manipulatives for learning and use research to guide them.

The findings found that the most relevant use of manipulatives in early childhood mathematics instruction identified four general principles:

(a) use a manipulative consistently, over a long period of time;
(b) begin with highly transparent concrete representations and move to more abstract representations over time;
(c) avoid manipulatives that resemble everyday objects or have distracting irrelevant features; and
(d) explicitly explain the relation between the manipulatives and the mathematical concept.

Cognitive science research suggests that instruction that follows these principles when using manipulatives is likely to lead to greater mathematics learning than instruction that does not.

Kirschner and Sweller

"The emphasis on the practical application of what is being learned seems very positive. However, it may be an error to assume that the pedagogic content of the learning experience is identical to the methods and processes (i.e., the epistemology) of the discipline being studied and a mistake to assume that instruction should exclusively focus on application." (p. 84).

It is therefore important to ensure that during teaching staff need to ensure pupils:

The four part model promotes creates a structure/sequence to follow within lessons.

Research – Counter Argument

Is It What We Think It Is? (Marshall and Swan 2008)

Seemingly there is an unchallenged assumption that the use of manipulative materials in the teaching of mathematics is a key to learning about mathematics concepts, and this is supported by the mathematics manipulatives industry.

Teachers who are time poor and under pressure are either looking for a miracle manipulative to solve all their problems with teaching a particular topic such as fractions or have abandoned the use of manipulatives in favour of textbooks.

In the sixties, the use of mathematics manipulatives was often justified on the basis of the ancient proverb:

I hear and I forget.

I see and I remember.

I do and I understand.

This proverb is still being used as a justification for the use of mathematics manipulatives.

Is It What We Think It Is? (Ball 1992)

In mathematics manipulatives without questioning the appropriateness or value of the manipulative being used. There tends to be a belief that mathematics manipulatives are inherently good and that lessons that make use of mathematics manipulatives are pedagogically sound.

Throughout this paper, the argument has been made that manipulatives on their own do not teach – teachers do. Children can often look very busy (active) with manipulatives but that does not necessarily mean that children are learning. Clements (1999) noted that simply using manipulatives as part of a mathematics lesson does not guarantee success.

Their physicality does not carry the meaning of the mathematical idea. They can be used in a rote manner ... They need teachers who can reflect on their students' representations for mathematical ideas and help them develop increasing sophisticated and mathematical representations (p. 3).

Research Audit

Why They Can Hinder Learning and What You Can Do About It (Fulmer 2017)

To use a manipulative effectively, children must be able to think about the manipulative in two ways: as an object (e.g., "this is a plastic teddy bear") and as a symbol (e.g., "this teddy bear represents one unit"). This ability, called dual representation, develops during early childhood. If an object is visually interesting or realistic, or if learners are already familiar with the object, they have more difficulty viewing that object in terms of a new symbolic meaning.

Research suggests that manipulatives can improve immediate learning outcomes (e.g., retention, recognition) but may **impede transfer of learning**. Students who learn with manipulatives can become too reliant on the object and context, and as a result, have difficulty transferring their knowledge to new contexts, different testing formats, or to abstract representations of the problem.

How can I use manipulatives effectively in my teaching?

- As educators, we need to carefully consider the manipulative, our learners, and the pedagogy to ensure that students' work with manipulatives supports their learning. Three recommendations are offered below:
- Select the right manipulative for the task and your students.
- Structure the environment for effective learning.
- Support transfer from concrete to abstract.

Select the right manipulative for the task and your students.

Ask yourself the following questions:

- What do you want students to know or be able to do by the end of the lesson?

- What is the role or purpose of the manipulative within the lesson?
- What features must the manipulative have to support learning of the target concept?
- What features of the manipulative might distract learners or lead them astray?
- What prior knowledge or experience might students have with this manipulative, and how could that impact their learning?

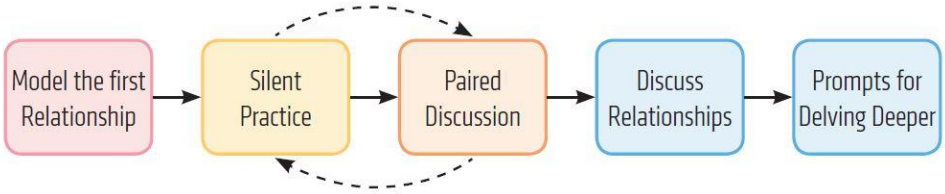
Structure the environment for effective learning.
Ask yourself the following questions:

- How will I communicate the purpose of the manipulative to students?
- How will I help students focus their attention on the relevant features of the manipulative?
- What prior knowledge do students need to use the manipulative effectively to understand the concept?
- How will I manage off-task behaviour? (Tip: don't allow students to use manipulatives in non-symbolic ways before or during the lesson)
- How will I model and scaffold use of the manipulative for students?

Support transfer from concrete to abstract.
We want students to be able to apply their knowledge and skills to new problems and contexts. This is called transfer. The practice of concreteness fading is effective in helping learners move from an understanding based on concrete objects to a more abstract understanding where they can solve problems without using the concrete objects. This teaching approach creates a middle step between the concrete and abstract, helping learners with the transfer process by making the process gradual and explicit.

Teaching Strategies
Intelligent Practice sequences?
Sequences of questions which enable pupils to gain practice in carrying out a mathematical method, whilst at the same time providing opportunities to think mathematically.

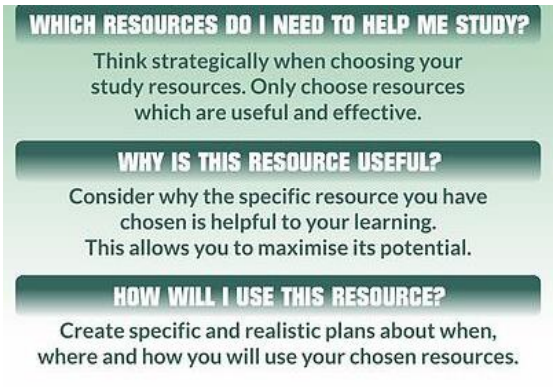
Process



The intention is to build this structure into lessons, teachers model/scaffold this process and pupils apply this to their manipulative experience.

A Simple Way to Improve Metacognition
Adapting this model would provide pupils with a mechanism to assess the need within the lesson regarding which manipulative would be needed but also if a manipulative would be needed. Pupils would go through a process of identification to assess the 'perceptual richness and efficacy' of the tool. The idea would be to adapt the questions to meet the needs of a manipulative approach.
Researchers found that having students ask themselves 3 simple questions can make a drastic difference: First question - *"Which resources do I need to help me study?" (What manipulatives do I need to help me learn?)*
Second question encourages students to think strategically when selecting study resources. This ensures that only useful and effective study resources are used as part of the revision process. *"Why are these resources useful?" (Why are these manipulatives useful?)*

Final question encourages students to consider why a specific resource is helpful to their learning, as well as getting students to think about how to maximise the potential of that resource. *"How will I use this resource?" (How will I use this manipulative?)*



Conclusion

Manipulatives are important to pupil success in schools but it is widely acknowledged by varying experts that certain criterion need to be applied if they are used 'widely and effectively (Ofsted)' in lessons. Consistency was a key factor in research studied alongside creating 'experts', staff and pupils understanding the function of the manipulative both in limitation and application across numerous mathematical concepts.

Within the work with teachers, there became a need to create 'Stop' points in the curriculum, times when the manipulative itself needed to be taught to the pupil regardless of the programme of study, it was deemed necessary to provide the pupil with expertise in the 'tool' (J. Back) before applying it to the concept being taught for that year group.

Transition or transfer points were important (Fuller) to ensure the pupil understand how the manipulative (concreteness) needed to be adapted to ensure transfer occurred. Providing the pupil with the skill set to practice and understand the transition between Concrete, Pictorial (Representational) and Abstract would be essential for success to be obtained.

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Tables

The project was focused upon identifying manipulative that were consistent across Programmes of Study, were able to be implemented across numerous Year Groups and were chosen based on their structure to support the objectives in the National curriculum 2014 rather than to a scheme currently employed by schools. The tables are simply the first stage in the process, PD training would need to be incorporated, Stop points and transition/transfer opportunities would need to be created and finally, adaptations may occur to ensure that decisions made 'on paper' are in fact changes in policy, practice and impact positively on pupils understanding of the tools they need to succeed in class.

Addition and Subtraction with Manipulatives - Progression in Mathematics Y1-6

National Curriculum for Mathematics

Addition and subtractions

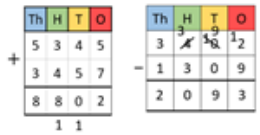
Year 1	Year 2	Year 3	Year 4	Year 5	Year 6
Pupils should be taught to:	Pupils should be taught to:	Pupils should be taught to:	Pupils should be taught to:	Pupils should be taught to:	Pupils should be taught to:

• Read, write and interpret mathematical statements involving addition (+), subtraction (-), and equals (=) signs
• Represent and use number bonds and related subtraction facts within 20.
• Add and subtract one-digit and two-digit numbers to 20, including zero.
• Solve one-step problems that involve addition and subtraction, using concrete objects and pictorial representations, and missing number problems such as 7 =□ - 9.
	• Solve problems with addition and subtraction:
	  ◦ Using concrete objects and pictorial representations, including those involving numbers, quantities and measures
	  ◦ Applying their increasing knowledge of mental and written methods
	• Recall and use addition and subtraction facts to 20 fluently and derive and use related facts up to 100.
	• Add and subtract numbers using concrete objects, pictorial representations, and mentally, including:
	• A two-digit number and ones
	• A two-digit number and tens
	• Two two-digit numbers
	• Adding three one-digit numbers
	• Show that addition of two numbers can be done in any order (commutative) and subtraction of one number from another cannot.
	• Recognise and use the inverse relationship between addition and subtraction and use this to check calculations and missing number problems.
		• Add and subtract numbers mentally, including:
		  ◦ A three-digit number and ones
		  ◦ A three-digit number and tens
		  ◦ A three-digit number and hundreds
		• Add and subtract numbers with up to three digits, using the efficient written methods of columnar addition and subtraction.
		• Estimate the answer to a calculation and use inverse operations to check answers.
		• Solve problems, including missing number problems, using number facts, place value, and more complex addition and subtraction.
			• Add and subtract numbers with up to 4 digits using the formal written methods of columnar addition and subtraction where appropriate.
			• Estimate and use inverse operations to check answers to a calculation.
			• Solve addition and subtraction two-step problems in contexts, deciding which operations and methods to use and why.
				• Add and subtract whole numbers with more than 4 digits, including using formal written methods (columnar addition and subtraction
				• Add and subtract numbers mentally with increasingly large numbers.
				• Use rounding to check answers to calculations and determine, in the context of a problem, levels of accuracy.
				• Solve addition and subtraction multi-step problems in contexts, deciding which operations and methods to use and why
					• Solve addition and subtraction multi-step problems in contexts, deciding which operations and methods to use and why.

Manipulatives

Concrete and Pictorial

Abstract



Interactive

Video (How to…)	Dienes/PV Grid and Counters Tens Frame	STOP Points Transition/Transfer Opportunities

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Fractions with Manipulatives - Progression in Mathematics Y1-6

National Curriculum for Mathematics

Fractions, Decimals and Percentages

Year 1

Pupils should be taught to:

- Recognise, find and name a half as one of two equal parts of an object, shape or quantity
- Recognise, find and name a quarter as one of four equal parts of an object, shape or quantity.

Year 2

Pupils should be taught to:

- Recognise, find name and write fractions 1/3, ¼, 2/4, and ¾ of a length, shape, set of objects or quantity.
- Write simple fractions e.g. ⅙ of 6 = 3 and recognise the equivalent of two quarters and one half

Year 3

Pupils should be taught to:

- Count up and down in tenths; recognise that tenths arise from dividing an object into 10 equal parts and in dividing one-digit numbers or quantities by 10
- Recognise, find and write fractions of a discrete set of objects; unit fractions and non-unit fractions with small denominators.
- Recognise and use fractions as numbers; unit fractions and non-unit fractions with small denominators.
- Recognise and show, using diagrams, equivalent fractions with small denominators.
- Add and subtract fractions with the same denominator within one whole (e.g. 5/7 + 1/7 = 6/7)
- Compare and order unit fractions with the same denominator.
- Solve problems that involve all of the above

Year 4

Pupils should be taught to:

- Recognise and show, using diagrams, families of common equivalent fractions.
- Count up and down in hundredths; recognise that hundredths arise when dividing an object by a hundred and dividing tenths by ten.
- Solve problems involving increasingly harder fractions to calculate quantities, including non-unit fractions where the answer is a whole number.
- Add and subtract fractions with the same denominator.
- Recognise and write decimal equivalents to ¼; ½; ¾
- Find the effect of dividing a one or two-digit number by 10 and 100, identifying the value of the digits in the answer as ones, tenths and hundredths.
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Year 5

Pupils should be taught to:

- Compare and order fractions whose denominators are all multiples of the same number.
- Identify, name and write equivalent fractions of a given fraction, represented visually, including tenths and hundredths.
- Recognise mixed numbers and improper fractions and convert from one to the other and write mathematical statements >1 as a mixed number (e.g. 2/5 + 4/5 = 6/5 = 1 1/5)
- Add and subtract fractions with the same denominator and denominators that are multiples of the same number.
- Multiply proper fractions and mixed numbers by whole numbers, supported by materials and diagrams.
- Read and write decimal numbers as fractions (e.g. 0.71 = 71/100)
- Solve problems which require knowing percentage and decimal equivalents of ½, ¼, 1/5, 2/5, 4/5 and those fractions with a denominator of a multiple of 10 or 25

Year 6

Pupils should be taught to:

- Use common factors to simplify fractions; use common multiples to express fractions in the same denomination.
- Compare and order fractions including fractions >1
- Add and subtract fractions with different denominators and mixed numbers, using the concept of equivalent fractions.
- Multiply simple pairs of proper fractions, writing the answer in its simplest form (e.g. ¼ x ½ = 1/8)
- Divide proper fractions by whole numbers (e.g. 1/3 ÷ 2 = 1/6)
- Associate a fraction with division and calculate decimal fraction equivalents (e.g. 0.375) for a simple fraction (e.g. 3/8)
- Solve problems which require answers to be rounded to specified degrees of accuracy.
- Recall and use equivalences between simple fractions, decimals and percentages, including in different contexts

Concrete	Pictorial	Abstract
Manipulatives			¾ + ½ = ¾ x ½ =
Interactive	Video (How to…)	Numicon Fractions App	STOP Points Transition/Transfer Opportunities

Multiplication and Division with Manipulatives - Progression in Mathematics Y1-6

National Curriculum for Mathematics

Multiplication and division

Year 1

Pupils should be taught to:

- Solve simple one step problems involving multiplication and division, by calculating the answer using concrete objects, pictorial representations and arrays with the support of the teacher.

Year 2

Pupils should be taught to:

- Recall and use multiplication and division facts for the 2, 5 and 10 multiplication tables, including recognising odd and even numbers.
- Calculate mathematical statements for multiplication and division within the multiplication tables and write them using the multiplication (x), division (÷) and equals (=) signs
- Show that multiplications of two numbers can be done in any order (commutative) and division of one number by another cannot.
- Solve one-step problems involving multiplication and division, using materials arrays, repeated addition, mental methods, and multiplication and division facts, including problems in contexts.

Year 3

Pupils should be taught to:

- Recall and use multiplication and division facts for the 3, 4 and 8 multiplication tables.
- Write and calculate mathematical statements for multiplication and division using the multiplication tables that they know, including for two-digit numbers times one-digit numbers, using mental and progressing to efficient written methods
- Solve problems, including missing number problems, involving multiplication and division, including integer scaling problems and correspondence problems in which n objects are connected to m objects.

Year 4

Pupils should be taught to:

- Recall multiplication and division facts for multiplication tables up to 12 x 12.
- Use place value, known and derived facts to multiply and divide mentally, including: multiplying by 0 and 1; dividing by 1; multiplying together three numbers.
- Recognise and use factor pairs and commutatively in mental calculations.
- Multiply two-digit and three-digit numbers by a one-digit number using formal written layout.
- Solve problems involving multiplying and adding, including using the distributive law to multiply two digit numbers by one digit, integer scaling problems and harder correspondence problems such as which n objects are connected to m objects.

Year 5

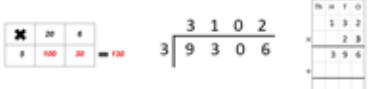
Pupils should be taught to:

- Identify multiples and factors, including finding all factor pairs of a number, and common factors of two numbers.
- Know and use the vocabulary of prime numbers, prime factors and composite (non-prime) numbers.
- Establish whether a number up to 100 is prime and recall prime numbers up to 19
- Multiply numbers up to 4 digits by a one- or two-digit number using a formal written method, including long multiplication for two-digit numbers.
- Multiply and divide numbers mentally drawing upon known facts.
- Divide numbers up to 4 digits by a one-digit number using the formal written method of short division and interpret remainders appropriately for the context.
- Multiply and divide whole numbers and those involving decimals by 10, 100 and 1000.
- Recognise and use square numbers and cube numbers, and the notation for squared (²) and cubed (³)
- Solve problems involving multiplication and division including using their knowledge of factors and multiples, squares and cubes.
- Solve problems involving addition, subtraction, multiplication and division and a combination of these, including understanding the meaning of the equals sign.
- Solve problems involving multiplication and division, including scaling by simple fractions and problems involving simple rates

Year 6

Pupils should be taught to:

- Multiply multi-digit numbers up to 4 digits by a two-digit whole number using the efficient written method of long multiplication.
- Divide numbers up to 4 digits by a two-digit whole number using the formal written method of long division, and interpret remainders as whole number remainders, fractions, or by rounding, as appropriate for the context.
- Divide numbers up to 4 digits by a two-digit number using the formal written method of short division where appropriate, interpreting remainders according to context.
- Perform mental calculations, including with mixed operations and large numbers.
- Identify common factors, common multiples and prime numbers.
- Use their knowledge of the order of operations to carry out calculations involving the four operations.
- Using their knowledge of the order of operations to carry out calculations involving the four operations.
- Solve problems involving addition, subtraction, multiplication and division.
- Use estimation to check answers to calculations and determine, in the context of a problem, levels of accuracy.

Concrete and Pictorial		Abstract
Manipulatives		Experience of partitioning calculating and array 7 x 6 = 24 ÷ 3 = 
Interactive	Video (How to…)	Tables/Hundred Squares, Place Value Sliders, Arrays	STOP Points Transition/Transfer Opportunities