

# Core Mathematics

## Induction Pack

### Introduction

Well done choosing Core Maths. It is a great way to keep your valuable maths skills up to date in preparation for employment or university study if you are not planning to take A level Mathematics. It will also help to support your work in other numerate subjects such as business, economics, geography, psychology, biology and chemistry. Finally, studying Core Maths will better equip you to apply calculations and statistics to problems in everyday life, including interpretation of data provided by the media.

The better prepared you are for Level 3 mathematics, the more you will gain from your advanced study. Your teacher will give you access to an online course called the Core Maths Skills course which you could use over the summer to make sure you are well-prepared. If you do not get a username through your school for any reason, you can register as an individual here: [https://my.integralmaths.org/integral/management/self\\_reg\\_students.php](https://my.integralmaths.org/integral/management/self_reg_students.php)

## Percentages

### One number as a percentage of another

It is often useful to convert a quantity to a percentage to get an idea of its size or to compare it to other numbers. It is vital to report what it is a percentage of.

For example, if you get 67 on a test with 80 marks, your percentage is

$$\frac{67}{80} = 0.8375 = 83.75\% \text{ which we would normally round to } 84\%$$

This is equivalent to  $67 \div 80$  on a calculator and then  $\times 100$  to make it appear as a percentage.

In a test it is obvious that you are talking about the percentage correct out of the total possible marks, but other situations are less straightforward.

	Wear glasses	Do not wear glasses
Year 12	12	54
Year 13	18	46

What percentage of Year 12 students wear glasses?

$$\frac{12}{12 + 54} = \frac{12}{66} = 0.18 = 18\%$$

What percentage of the students who wear glasses are Year 12?

$$\frac{12}{12 + 18} = \frac{12}{30} = 0.4 = 40\%$$

The questions sound similar, but the answers are very different. Always check that you know what you are finding a percentage of.

### Percentage change

When you are finding the percentage change from one value to another, divide the difference by the original amount.

For example, if the population of a village increased from 234 to 275 during one year, what was the percentage increase?

$$275 - 234 = 41 \quad \frac{41}{234} = 0.17521 \dots = 17.5\%$$

### Finding a percentage of an amount

Change the percentage to a fraction or a decimal and multiply.

For example, 35% of £120 is  $\frac{35}{100} \times 120 = £42$  or  $0.35 \times 120 = £42$ .

Notice the fraction method is equivalent to dividing by 100 to find 1% and then multiplying by 35 to get 35%. You can also use equivalent fractions, so that you can find 25% by dividing by 4 (as 25% is  $\frac{1}{4}$ ) and 10% by dividing by 10 (as 10% is  $\frac{1}{10}$ ). It's a good idea to have non-calculator methods like this up your sleeve for quick calculations and for checking your answers.

### Finding a percentage increase or decrease

You can just use the method above to find the percentage of the original amount and then add it or subtract it as necessary.

For example, to increase £120 by 35%, you could find 35% of £120 as above and add it on: £120 + £42 = £162. (Decreasing by the same percentage would just mean subtracting instead of adding.)

It's really helpful if you can do this in just one step though.

If you want to increase £120 by 35%, then you want 135% of £120.

This is just  $1.35 \times 120 = 162$ .

If you decrease £120 by 35%, you want 65% of £120. So  $0.65 \times 120 = 78$ .

This is especially useful if you want to repeat the same percentage increase or decrease over and over, as in compound interest (repeated increase) or depreciation (repeated decrease).

### Finding the original amount

When an amount has been increased or decreased by a particular percentage, and we only know the new amount, it can be tricky to get back to the original.

For example, if the price of a coat has been reduced by 20% in a sale and now costs £48, it can be tempting to think that we could just increase the £48 by 20% to get back to the original. THIS WILL NOT WORK. This is because the discount is 20% of the original price, not 20% of the new price and they are different numbers.

Instead, remember that if 20% was taken off the price, 80% remains.

So 80% = £48. We need to work out what 100% of the original price was and there are lots of ways to do this. The simplest is probably dividing by 80 to get 1% and then multiplying by 100 to get 100%. But you could also divide by 8 to get 10% and multiply by 10 or divide by 4 to get 20% and multiply by 5.

$$£48 \div 80 = 0.6 \quad 0.6 \times 100 = £60$$

$$£48 \div 8 = 6 \quad 6 \times 10 = £60$$

$$£48 \div 4 = 12 \quad 12 \times 5 = £60$$

## Exercise A

1. A school has 80 staff. 15% of the staff wear glasses. Calculate the number of staff that wear glasses.
2. Ayesha plays hockey. Last year Ayesha scored 8 goals. This year Ayesha scored 13 goals. Calculate the percentage increase in the number of goals scored.
3. Between 2001 and 2011, the population of a town increased by 8%. In 2001 the population was 34 342. Calculate the population in 2011.
4. In a sale, normal prices were reduced by 20%. The normal price of a camera was £180. Work out the sale price of the camera.
5. Justin bought some clothes. The clothes should have cost £84.00 but he got a discount of 15%. Work out how much money Justin saved.
6. Susanna invested £2000 for 3 years at 4% interest per annum compound interest. Work out the amount of interest Susanna had earned after 3 years.
7. Anya bought a car for £12 500. The car depreciates at a rate of 12% per year. Work out the value of the car after five years.
8. The price of shoes was increased by 15%. However, customers were given a 20% discount if they bought two pairs at the same time. Work out the cost of two pairs of shoes that originally cost £68 each.
9. The price of a new TV is £540, which includes 20% VAT. Find the cost of the TV excluding VAT.
10. Extension: Natasha invested some money at 4% per annum compound interest. At the end of two years, the value of her investment was £3380. Find the amount of money that Natasha invested.

## Mean, median, mode and range

The mean, median, and mode are different types of average. We use them when we have a set of numbers, and we want to get a feel for how big/small that group of numbers is. Are they very high? Are they only a little bigger than zero? Are they closer to 10 or 20 in general? These are the sort of questions we're answering when finding an average.

### Mean

The mean is the most frequently used average. To find the mean we must add up all the numbers we're finding the average of, and then divide by how many numbers there are in that list.

### Median

The median is often referred to as "the middle", which is precisely what it is. To find the median of a list of numbers, we put the numbers in order from smallest to largest and find the middle value/middle two values. If there is a middle value, then that is the median; if there are two middle values, then the median is the halfway point between the two.

There are two common ways of finding the middle value(s):

Cross out the smallest number and the largest number, then cross out the next smallest and largest, keeping going crossing out pairs of number like this until you have one or two left.

If  $n$  is the number of values in the list, then work out the value of  $\frac{n+1}{2}$ . If the answer is a whole number such as 5, then the median is the 5th point along the ordered list. If the answer is a decimal (ending in .5) such as 12.5, then the median is the halfway point between the 12th and 13th value along.

### Mode

The mode is the most common value. To find it, look for which value appears most often. Note: there might be two values which are tied for the most appearances, in which case we say the data is bimodal, or alternatively there might be no repeats at all, in which case there is simply no mode.

Example: 9 people take a test. Their scores out of 100 are:

56, 79, 77, 48, 90, 68, 79, 92, 71

Work out the mean, median, and mode of their scores.

First up, the mean. The question tells that there are 9 data points, so we must add the numbers together and divide the result by 9.

$$\frac{56+79+77+48+90+68+79+92+71}{9} = 73.3 \text{ (1 dp)}$$

So, the mean is 73.3.

Next up, the median. Firstly, we have to put the numbers in ascending order. This looks like

48, 56, 68, 71, 77, 79, 79, 90, 92

There are 9 numbers, and  $\frac{9+1}{2} = 5$ , so the median must be the 5th term along. Counting along the list, we get that the median is 77. Finally, the mode. We can see very clearly from the ordered list that there is only one repeat: 79, so the mode is 79.

It's important to understand that the mean, median, and mode all come with their advantages and disadvantages. To understand these, you need to know another definition – outliers. An outlier is a piece of data that doesn't quite fit with the rest of them. For example, in the list

500, 520, 493, 480, 530, 7

7 is an outlier. Outliers can occur for all kinds of reasons and often it's just human error. Now we know what an outlier is, let's see what's good and what's not about the mean, median, and mode.

Mean:

Advantage – every bit of data is used in calculating the mean, so it represents all the data. The mean is generally considered the better of the 3 averages, but it certainly isn't always.

Disadvantage – it is highly affected by outliers. The fact that every bit of data is used to calculate the mean can be a weakness. The list we just saw: 500, 520, 493, 480, 530, 7– what is its mean? Is it representative of where the data is? How about if you take the 7 out and then find the mean, is that a better average?

Median:

Advantage – it is not affected by outliers.

Disadvantage – it does not consider all the data. Consider the values 1, 1, 2, 3, 12, 14, 15– what is the median? Does it actually represent the middle of these numbers?

Mode:

Advantage – it is not affected by outliers.

Disadvantage(s) – firstly, it sometimes is impossible to find. Secondly, it does not consider all of the data. Consider the values 32, 35, 35, 128, 201, 176, 295– what is the mode? Does it represent the “average” of the data?

## Range

The range is not another average – it is a measure of spread. This means the range is a way of telling us how spread out the data is. To calculate it, we subtract the smallest value from the biggest value.

Example: Find the range of 12, 8, 4, 16, 15, 15, 5, 15, 10, 8.

A good way to make sure you haven't missed any numbers in determining the biggest and smallest value is to order them. Doing this, we get

4, 5, 8, 8, 10, 12, 15, 15, 15, 16.

Therefore, the smallest value is 4 and the largest is 16 and  $16 - 4 = 12$ , so the range is 12.

Sadly, the range also has its disadvantages – it is highly affected by outliers, just like the mean. If I included the value 100 in the list above, suddenly the range would be  $100 - 4 = 96$ , which isn't really a fair description of how spread out the data are since the rest are quite close together.

## Exercise B

1. Find the mode and range of the list of number below.

280, 350, 320, 400, 350, 490, 590, 470, 280, 410, 350

2. Dani recorded the heights of members of her extended family, to the nearest cm. Find the median of their heights (listed below).

163, 185, 164, 170, 188, 154, 168, 179

3. Below is a list of recorded reaction times in seconds.

0.25, 0.34, 0.39, 0.38, 0.39, 1.67, 0.28, 0.30, 0.42, 0.46

a) Calculate the mean of these values.

b) Identify any outliers and explain how the value of the mean would change if any outliers were to be removed from the calculation of the mean.

## Gradients of Straight Lines

The gradient of a line is a measure of how steep it is. If the gradient is small, the slope of the line will be very gradual, but if the gradient is big, the line will be quite steep. You should know how to calculate the gradient from two possible circumstances:

You are given the line drawn on a graph;

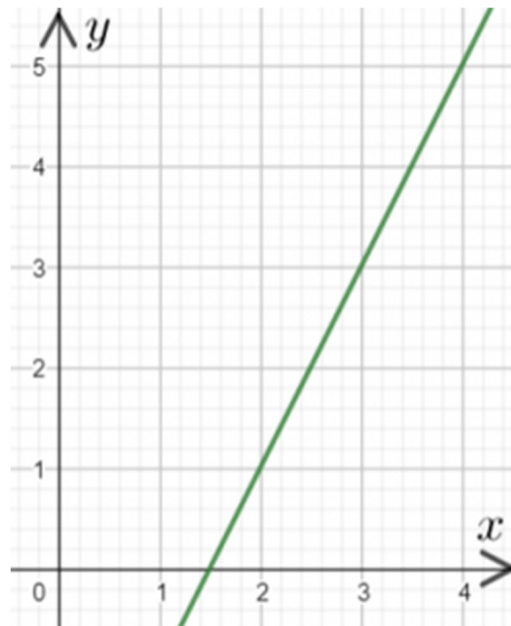
You are given two coordinates and told that a line passes through both of them.

These situations may seem distinct, but they aren't really. They both come down to the same idea:

$$\text{gradient} = \frac{\text{change in } y}{\text{change in } x}$$

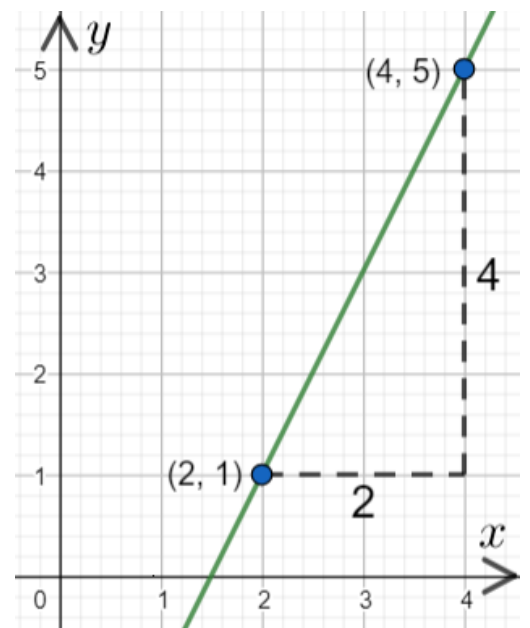
Additionally, the gradient is also a rate of change, specifically a rate of change of  $y$  with respect  $x$  – this is why we divide the change in  $y$  by the change in  $x$  to calculate it. What this all means is this: the gradient of a line tells us how much the  $y$  value changes when the  $x$  increases by 1. For example, if you have a line with gradient 3, then every time the  $x$  value increases by 1, the  $y$  value increases by 3.

Example: Calculate the gradient of the line drawn below.



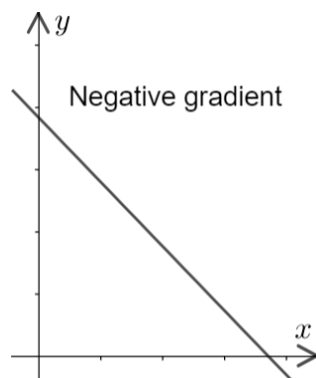
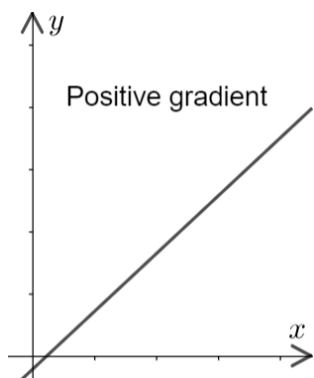
To do this, you want to pick 2 points on the graph that the line passes through. It's best, if you can, to pick two points where the coordinates are easy to read off.

Here, we picked (2, 1) and (4, 5), as seen on the graph on the right. Once you've done this, draw the right-angled triangle as pictured with dotted lines. Then, the change in x is the width of the base of that triangle, whilst the change in y is the height.



Therefore, we get gradient =  $\frac{4}{2} = 2$ .

You should be wary of the possibility of a negative gradient.



We just saw the positive gradient case. In the negative gradient case, the y value decreases as the x value increases. You can still use the triangle method seen above for working out gradient, but if the line is sloped downwards then you'll need to put a minus sign in front of your answer.

Example: Work out the gradient of the straight line that passes through (2, 3) and (-10, 6).

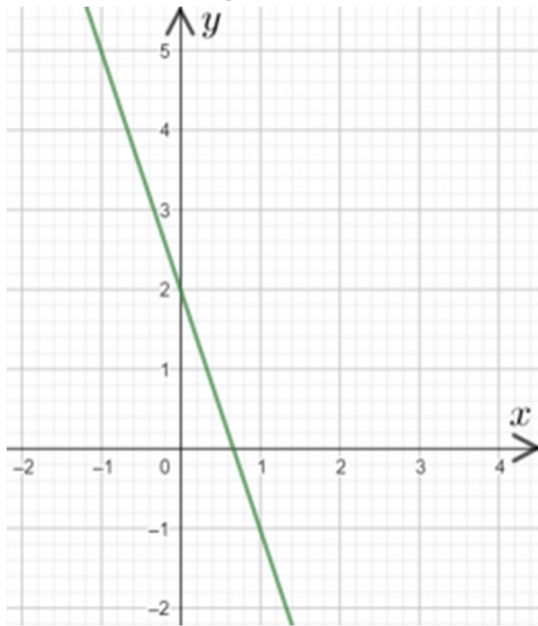


To find the change in  $x$  and change in  $y$  here, we must pick one of the points and subtract its  $x$  and  $y$  coordinates from the other point's  $x$  and  $y$  coordinates, respectively. Subtracting the first from the second, we get

$$\text{gradient} = \frac{6-3}{-10-2} = \frac{3}{-12} = -0.25$$

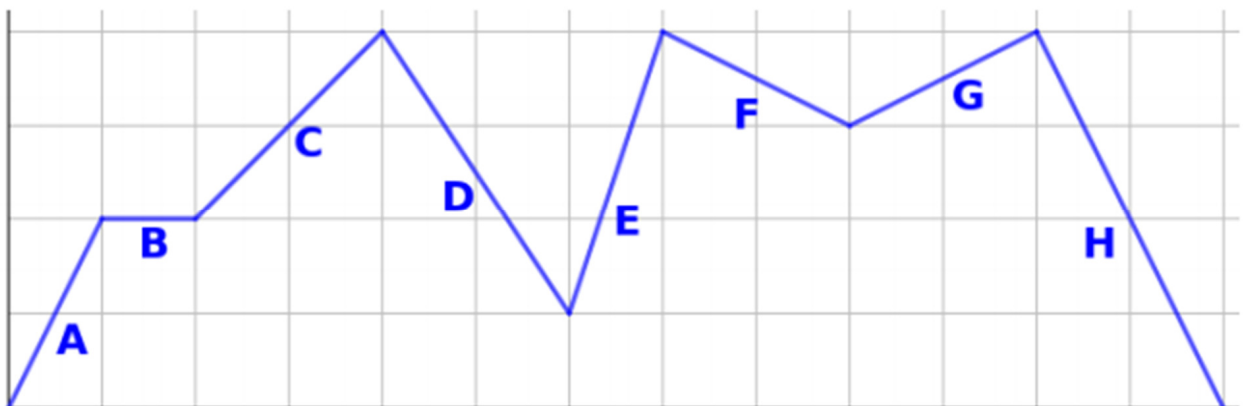
### Exercise C

1. Calculate the gradient of the line drawn below.



2. Find the gradient of the line that passes through  $(-8,-1)$  and  $(2,-6)$ .

3. Use the graph below to answer the following questions:



- a) Which section of the graph has the steepest positive gradient?
- b) Assuming the units on the  $x$ - and  $y$ -axes are the same, what is this gradient?
- c) For what section of the graph is  $y$  not changing (so the gradient is 0)?
- d) What section of the graph has the steepest negative gradient?
- e) What is the gradient of section D?

Check your answers to exercises A, B and C before you move on to the next section of the pack.

## Solutions

### Exercise A

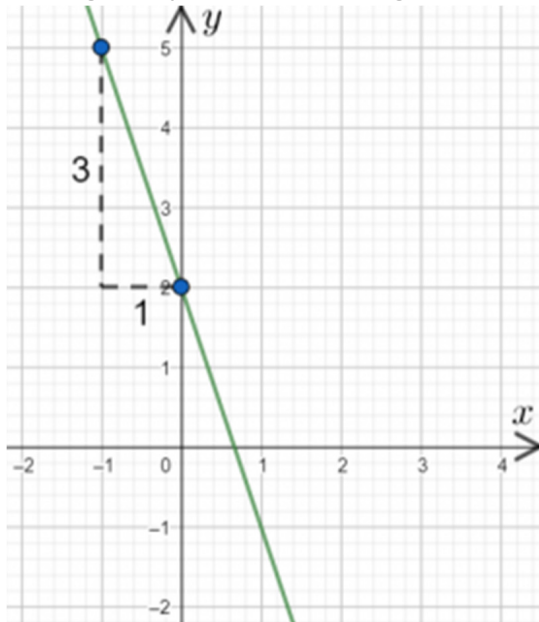
- $80 \times 0.15 = 12$
- $13 - 8 = 5$      $5 \div 8 = 0.625$      $0.625 = 62.5\%$
- $34\,342 \times 1.08 = 37\,089$
- $100\% - 20\% = 80\%$      $180 \times 0.8 = \text{£}144$
- $84 \times 0.15 = \text{£}12.60$
- $2000 \times 1.04^3 = 2249.73$      $2249.73 - 2000 = \text{£}249.73$
- $12\,500 \times 0.88^5 = \text{£}6596.65$
- $68 \times 2 = 136$      $136 \times 1.15 = 156.40$      $156.40 \times 0.8 = \text{£}125.12$
- $\text{£}540 = 120\%$      $540 \div 120 = 4.5$      $4.5 \times 100 = \text{£}450$
- $\text{£}3380 \div 1.04^2 = \text{£}3125$

### Exercise B

- The most common number is 350, so mode = 350. The lowest value is 280 and the highest is 590, so the range is  $590 - 280 = 310$ .
- Firstly, we must order them. This looks like  
154, 163, 164, 168, 170, 179, 185, 188  
There are 8 terms, and  $\frac{8+1}{2} = 4.5$ , so the median is halfway between the 4th term and the 5th term. The 4th term is 168 and then 5th term is 170, so the halfway point is 169.
- a) We need to add up all the values and divide by how many there are, which in this case is 10.  
$$\frac{0.25 + 0.34 + 0.39 + 0.38 + 0.39 + 1.67 + 0.28 + 0.30 + 0.42 + 0.46}{10} = 0.488$$
  
b) Clearly, 1.67 is much higher than the rest of the values so it is an outlier. If this outlier were removed, then the mean would be lower.

## Exercise C

1. We must find two points that the line passes through and draw a right-angled triangle underneath, so we can identify the change in  $x$  to be the base and the change in  $y$  to be the height. This looks like



(There are other points you could choose for your triangle, but the result should be the same.)

Now, given that this is a downwards slope, it must have a negative gradient. So, we

get  $\text{gradient} = -\frac{3}{1} = -3$ .

2.  $\text{Gradient} = \frac{-6 - (-1)}{2 - (-8)} = \frac{-5}{10} = -0.5$

3. a) E

b)  $\frac{3}{1} = 3$

c) B

d) H (the gradient is -2)

e)  $-\frac{3}{2} = -1.5$

## Climate Change in Numbers and Graphs

The remainder of this pack is about the issue of climate change. You need to apply your mathematical skills to interpret the information given and also think critically, expressing your opinions and judgements and using calculations to back these up where possible.

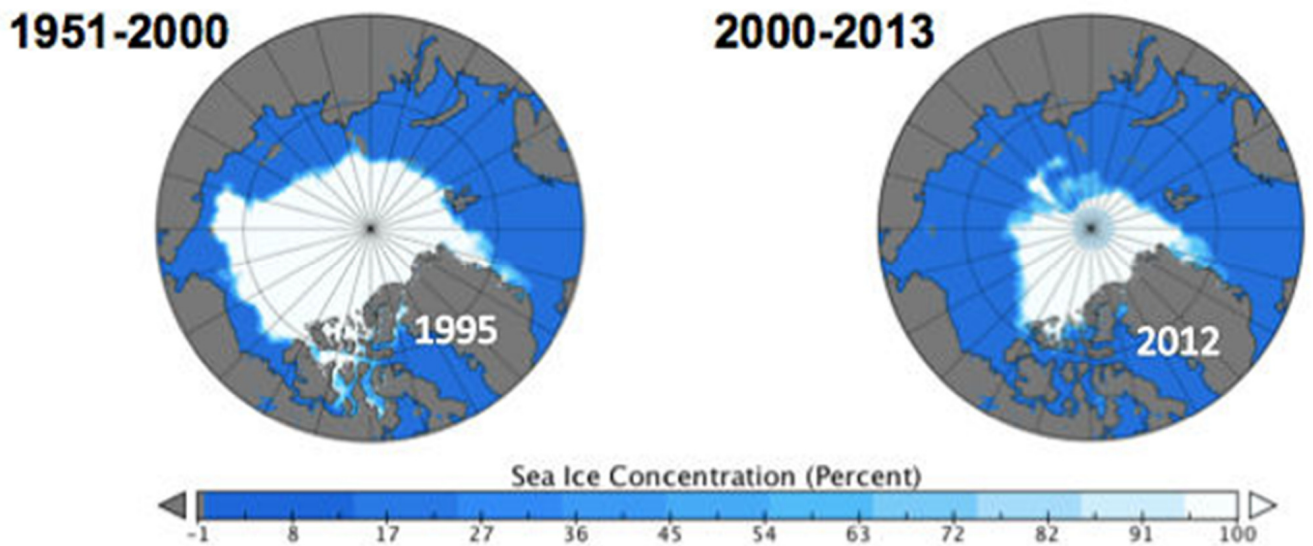
## Exercise D

1. According to a scientist at Stanford University, the early severe lockdowns in China in 2020 caused dramatic reductions in air pollution which could have saved between 50,000 and 75,000 lives.
  - a. Approximately what percentage is this of the 7 million people that the WHO estimates die from exposure to fine particles in polluted air each year?
  - b. Approximately what percentage is this of the population of China, estimated to be 1.4 billion in 2020?
  - c. How would you describe the comparison between the number of lives which may have been saved through better air quality to the number of COVID-related deaths in China, 4636? Please use words and calculations and feel free to research other data if you wish.
2. The graph shows satellite measurements of global sea level rise since 2000. (mm = millimetres)



- a. Draw a line of best fit through the data between 1993 and 2010, and work out the rate of global sea level rise.
- b. Draw the line of best fit through the data between 2011 and 2017, and work out the rate of global sea level rise.
- c. What does the difference between the rate in part a) and part b) tell us about how global sea level rise is changing?

3. The increasing global temperature due to human-induced climate change is causing ice in the Arctic to melt, particularly over the summer season, July to September.



In September 1999, there was 6.2 million km<sup>2</sup> of Arctic sea ice.

In September 2015, there was 4.6 million km<sup>2</sup> of Arctic sea ice.

Work out the percentage decrease in Arctic sea ice from September 1999 to September 2015.

4. One way of reducing carbon dioxide in the atmosphere and slowing global warming is to plant trees which absorb carbon dioxide from the atmosphere.

A scientist plants some trees in the Amazon rainforest and in the Tongass rainforest and determines that while the trees in the Amazon absorb 20% more carbon dioxide per tree, 25% fewer trees have been planted in the Amazon. How does the amount of carbon dioxide absorbed in the Amazon compare to the carbon dioxide absorbed in the Tongass rainforest?

5. Chocolate is made from a crop called cocoa, which grows on a tree. Cocoa crops could be affected by climate change since increased frequency and intensity of extreme weather means more cocoa trees do not survive.

The number of successful cocoa trees in a given area is modelled as

$C = 16500 \times 0.82^n$  where  $n$  is the number of years after the cocoa trees are first planted.

- Write down the number of trees that are first planted in this model.
- What is the annual percentage increase or decrease of successful cocoa trees?
- Show that the number of successful cocoa trees after 4 years is less than half the number of trees planted.

## Solutions to Exercise D

- 50,000 – 75,000 is between 0.7% and 1.1% of the estimated deaths from polluted air.
  - 50,000 – 75,000 is between 0.004% and 0.005% of the population of China.
  - 50,000 is more than 10 times the number of deaths from coronavirus in China. But perhaps this is not the most relevant comparison. One could argue that the severe early lockdowns saved the lives of people in China from coronavirus and air pollution and that many more could have died from both. Is it possible to estimate how many more would have died from coronavirus had the lockdowns not taken place?
- Roughly:  $\frac{51-4}{2010.5-1993} = 2.6857 = 2.69$  mm/year
  - Roughly:  $\frac{84-0}{2018-1999.5} = 4.5405 = 4.54$  mm/year
  - Global sea levels are rising at a faster and faster rate.
- $\frac{6.2-4.6}{6.2} \times 100 = 25.8\%$
- When the trees are first planted,  $n = 0$  so the number of trees is 16500.
  - Each year the number of trees is multiplied by 0.82 which is equivalent to an 18% decrease.
  - $16500 \times 0.82^4 = 7460$  trees.  $\frac{1}{2}$  of 16500 is 8250 so fewer than half the trees are successful after 4 years.