AS and A-level Further Mathematics

Pure Maths Teaching Guidance

AS (7366) and A-level 7367

For teaching from September 2017 onwards

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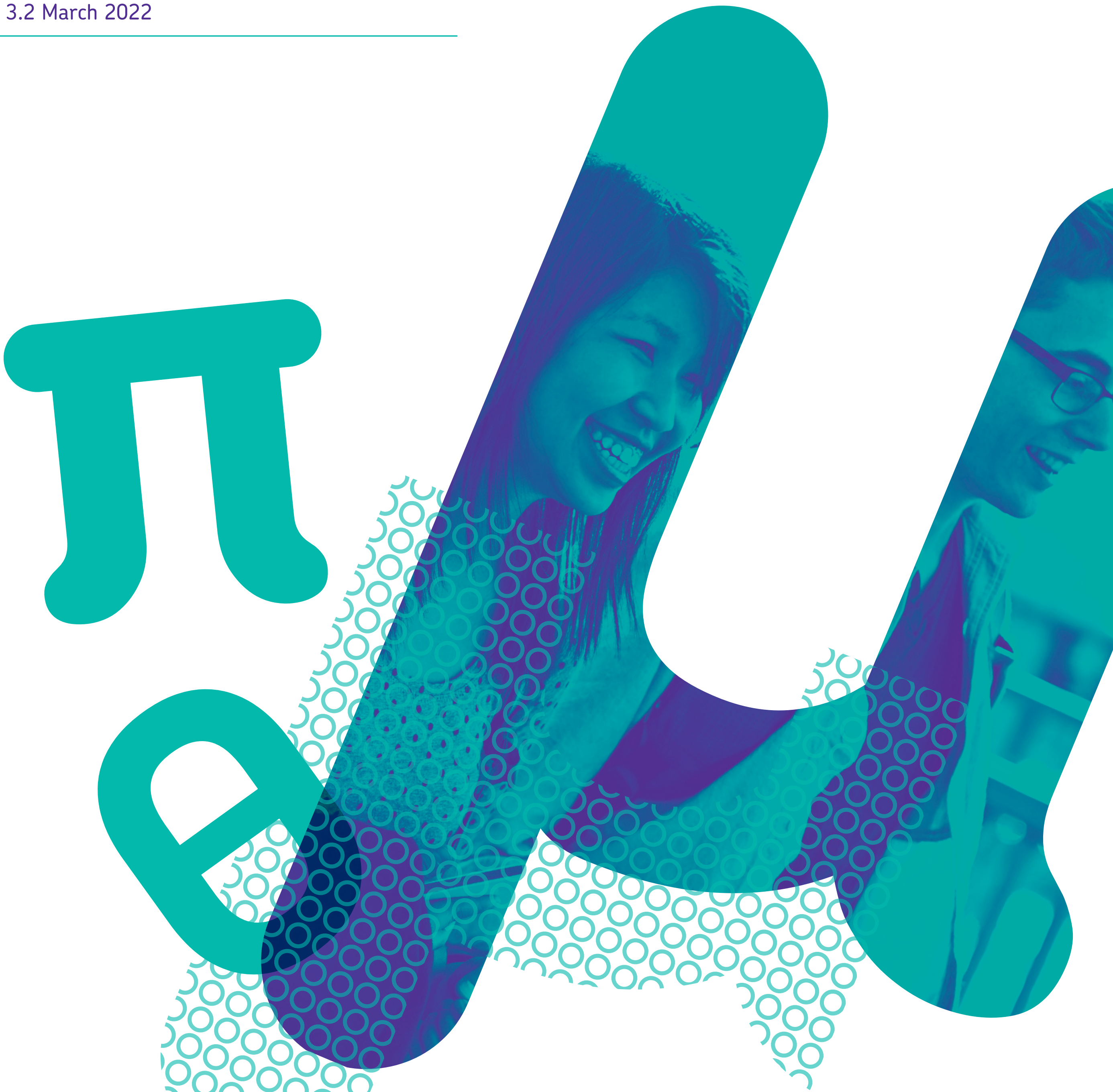

Our specification is published on our website (aqa.org.uk). We will let centres know in writing about any changes to the specification. We will also publish changes on our website. The definitive version of our specification will always be the one on our website and may differ from printed versions.

You can download a copy of this teaching guidance from our All About Maths website (allaboutmaths.aqa.org.uk/). This is where you will find the most up-to-date version, as well as information on version control.

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General information – disclaimer

This AS and A-level Further Mathematics teaching guidance will help you plan your teaching by further explaining how we have interpreted content of the specification and providing examples of how the content of the specification may be assessed. The teaching guidance notes do not always cover the whole content statement.

The examples included in this guidance have been chosen to illustrate the level at which this content will be assessed. The wording and format used in this guidance do not always represent how questions would appear in a question paper. Not all questions in this guidance have been through the same rigorous checking process as the ones used in our question papers.

Several questions have been taken from legacy specifications and therefore represent higher levels of AO1 than will be found in a suite of exam papers for this A-level Further Mathematics specification.

This guidance is not intended to restrict what can be assessed in the question papers based on the specification. Questions will be set in a variety of formats including both familiar and unfamiliar contexts.

All knowledge from the GCSE Mathematics specification is assumed.

Subject content

This Teaching guidance is designed to illustrate the detail within the content of the AS and A-level Further Mathematics specification.

Half the subject content was set out the Department for Education (DfE). The remaining half was defined by AQA, based on feedback from Higher Education and teachers.

Content in **bold type** is contained within the AS Further Mathematics qualification as well as the A-level Further Mathematics qualification. Content in standard type is contained only within the A-level Further Mathematics qualification.

A Proof

A1 Construct proofs using mathematical induction; contexts include sums of series, divisibility, and powers of matrices.

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- construct clear and concise arguments, making correct and appropriate use of symbols including $\Rightarrow$, $\Leftrightarrow$ and $\Leftarrow$
- explain steps in their reasoning or identify errors in a given argument
- understand the circumstances under which proof by induction can be used
- demonstrate the three steps required to complete proof by induction:

  1. Prove that $P_k \Rightarrow P_{k+1}$, where P_n is the statement to be proved.
  2. Prove that P_1 is true (or P_0 or some other starting value, depending on the question).
  3. Combine steps 1 and 2 to conclude the proof that P_n is true for all n.

Examples

1 A student attempts to use proof by induction to show that $n^2 - n$ is odd for all $n \in$

He argues as follows:

Assume true for $n = k$, where k is a positive integer

For $k + 1$,

$$\begin{aligned}(k+1)^2-(k+1) &= k^2+2k+1-k-1\\ &= k^2+k\\ &= k^2-k+2k\end{aligned}$$

Which must be odd, since $k^2 - k$ is assumed to be odd and $2k$ is even.

Therefore true for $k \Rightarrow$ true for $k + 1$.

Hence, by induction, $n^2 - n$ is always odd for positive integer values of n.

Explain the mistake in the student's proof.

2 (a) Show that $(k+1)(4(k+1)^2-1) = 4k^3 + 12k^2 + 11k + 3$

(b) Prove by induction that for all integers $n \geq 1$, $1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2 = \frac{1}{3}n(4n^2-1)$

3 The polynomial $\mathrm{p}(n)$ is given by $\mathrm{p}(n) = (n-1)^3 + n^3 + (n+1)^3$

Prove by induction that $\mathrm{p}(n)$ is a multiple of 9 for all integers $n \geq 1$.

4 If $\mathbf{M} = \begin{bmatrix} 2 & p \\ 0 & 1 \end{bmatrix}$, prove by induction that $\mathbf{M}^n = \begin{bmatrix} 2^n & (2^n-1)p \\ 0 & 1 \end{bmatrix}$

B Complex numbers

B1 Solve any quadratic equation with real coefficients; solve cubic or quartic equations with real coefficients (given sufficient information to deduce at least one root for cubics or at least one complex root or quadratic factor for quartics).

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- factorise, complete the square or use the quadratic formula to find complex roots of quadratics
- make appropriate use of calculators to find complex roots
- use the factor theorem with complex roots
- use any appropriate method to factorise cubics and quartics including inspection, comparing coefficients or polynomial division.

Examples

1 The polynomial $\mathrm{p}(x)$ is given by $\mathrm{p}(x) = x^2 + 2x + a$ where $a \in \mathbb{R}$

Find the smallest value of a so that $\mathrm{p}(x) = 0$ has complex roots.

2 Given that $3 + 2\mathrm{i}$ is a root of the equation $x^3 - 11x^2 + 43x + d = 0$, write $x^3 - 11x^2 + 43x + d$ as the product of a quadratic and a linear factor with real coefficients.

3 Given that $1 + \mathrm{i}$ is a root of the equation $x^4 + x^3 - 2x^2 + cx + d = 0$, where c and d are real, find the other three roots of the equation.

B2

Add, subtract, multiply and divide complex numbers in the form $x + \mathrm{i}y$ with x and y real; understand and use the terms 'real part' and 'imaginary part'.

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- use the fact that two complex numbers are equal if and only if their real parts are equal and their imaginary parts are equal
- use their calculators to perform calculations with complex numbers when appropriate
- collect real and imaginary parts
- manipulate complex numbers algebraically, simplifying results appropriately using $\mathrm{i}^2 = -1$.

Examples

1 Prove that the result of dividing a complex number by a non-zero complex number is always a complex number.

2 It is given that $z = \mathrm{i}(1 + \mathrm{i})(2 + \mathrm{i})$

Express z in the form $a + b\mathrm{i}$, where a and b are integers.

Note: in this question, it would be appropriate for a student to evaluate on a calculator.

3 $z = (2 + 3\mathrm{i})(7 + b\mathrm{i})$

Given that $\mathrm{Re}(z) = 0$, find the exact value of b.

AQA

B3

Understand and use the complex conjugate; know that non-real roots of polynomial equations with real coefficients occur in conjugate pairs.

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- use z^* to denote the complex conjugate of z
- understand that if $z = a + b\mathrm{i}$ then $z^* = a - b\mathrm{i}$
- state another root of a real polynomial equation given one non-real root.

Examples

1 The complex number $2 + 3\mathrm{i}$ is a root of the quadratic equation $x^2 + bx + c = 0$ where b and c are real numbers.

(a) Write down the other root of this equation.

(b) Find the values of b and c.

2 It is given that $z_1 = \frac{1}{2} - \mathrm{i}$

(a) (i) Calculate the value of z_1^2 giving your answer in the form $a + b\mathrm{i}$.

(ii) Hence verify that z_1 is a root of the equation $z^2 + z^* + \frac{1}{4} = 0$

(b) Show that $z_2 = \frac{1}{2} + \mathrm{i}$ also satisfies the equation in part (a)(ii).

(c) Show that the equation in part (a)(ii) has two equal real roots.

3 (a) It is given that $z = x + \mathrm{i}y$, where x and y are real numbers.

(b) Find, in terms of x and y, the real and imaginary parts of $(1 - 2\mathrm{i})z - z^*$

(c) Hence, find the complex number z such that $(1 - 2\mathrm{i})z - z^* = 10(2 + \mathrm{i})$

B4

Use and interpret Argand diagrams.

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- represent a complex number by a point on an Argand diagram
- understand geometrical properties of complex numbers on an Argand diagram, for example, that z^* is the reflection of z in the real axis.

Examples

1 Sketch an Argand diagram and clearly label the numbers i^7, i^8, i^9 and i^{10}

2 A square is drawn on an Argand diagram, with its centre at 0.

The corners of the square are labelled *A*, *B*, *C* and *D* in an anticlockwise sense.

Given that *A* is at the point which represents $-2-2\mathrm{i}$, state clearly the complex numbers that represent the positions of *B*, *C* and *D*.

B5

Convert between the Cartesian form and the modulus-argument form of a complex number (knowledge of radians is assumed).

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- understand the notation arg(z) for the argument and $|z|$ for the modulus
- convert between $z = a + b\text{i}$ and $z = r(\cos\theta + \text{i}\sin\theta)$
- use $r = \sqrt{a^2 + b^2}$ for the modulus
- use $\tan\theta = \dfrac{b}{a}$ where $-\pi < \theta \le \pi$ for the principal argument, θ
- understand the relationship between z and $z*$ in modulus-argument form.

Examples

1 $z = 3 - 4\text{i}$

Find arg(z), giving your answer to 2 decimal places.

2 $z = a + b\text{i}$ where $|z| = 10$ and $\arg(z) = \dfrac{2\pi}{3}$

Find the exact values of a and b.

3 $z = 2 - 2\text{i}$

(a) Write z in the form $r(\cos\theta + \text{i}\sin\theta)$ giving r and θ in an exact form.

(b) Write z^* in the form $r(\cos\theta + \text{i}\sin\theta)$ giving r and θ in an exact form.

B6

Multiply and divide complex numbers in modulus-argument form (knowledge of radians and compound angle formulae is assumed).

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- understand and use $|z_1 z_2| = |z_1||z_2|$ and $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$
- understand and use $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$ and $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$
- prove the results above.

Notes

The proofs referred to above use the compound angle formulae, which are not required in AS Maths. Similarly the use of radians is expected for complex numbers although radians are not required in AS Maths. Students studying AS Further Maths before the A-level content of Maths will need to be introduced to these two topics.

Examples

1 Find $\frac{z_1}{z_2}$ if $z_1 = 2\left(\cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3}\right)$ and $z_2 = 3\left(\cos\frac{\pi}{6} - \mathrm{i}\sin\frac{\pi}{6}\right)$

2 $z_1 = p(\cos A + \mathrm{i}\sin A)$ and $z_2 = q(\cos B + \mathrm{i}\sin B)$

Prove that $z_1 z_2 = pq(\cos(A+B) + \mathrm{i}\sin(A+B))$

3 The complex number z is defined by $z = \dfrac{1+3\mathrm{i}}{1-2\mathrm{i}}$

(a) (i) Express z in the form $a + \mathrm{i}b$.

(ii) Find the modulus and argument of z, giving your answer for the argument in the form $p\pi$ where $-1 < p \le 1$.

(b) The complex number z_1 has modulus $2\sqrt{2}$ and argument $-\dfrac{7\pi}{12}$

The complex number z_2 is defined by $z_2 = zz_1$.

(i) Show that $|z_2| = 4$ and $\arg z_2 = \dfrac{\pi}{6}$

(ii) Mark on an Argand diagram the points P_1 and P_2 that represent z_1 and z_2 respectively.

(iii) Find, in surd form, the distance between P_1 and P_2.

4 The complex numbers z and w are such that $z = (4 + 2\mathrm{i})(3 - \mathrm{i})$ and $w = \dfrac{4+2\mathrm{i}}{3-\mathrm{i}}$

(a) Express each of z and w in the form $a + \mathrm{i}b$ where a and b are real.

(b) (i) Write down the exact values of the modulus and argument of each of the complex numbers $4 + 2\mathrm{i}$ and $3 - \mathrm{i}$.

(ii) The points O, P and Q in the complex plane represent the complex numbers $0 + 0\mathrm{i}$, $4 + 2\mathrm{i}$ and $3 - \mathrm{i}$ respectively.

Find the exact length of PQ and hence, or otherwise, show that the triangle OPQ is right-angled.

B7

Construct and interpret simple loci in the Argand diagram such as $|z - a| < r$ and $\arg(z - a) = \theta$

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- construct loci of equations or inequalities involving circles $|z - z_1| = d$ or half lines $\arg(z - z_1) = \alpha$
- construct loci involving combinations of these
- use geometric properties to solve problems.

Notes

Loci of the form $|z - z_1| = |z - z_2|$ or $\arg(z - z_1) - \arg(z - z_2) = \theta$ are not required.

Examples

1 Sketch on an Argand diagrams the locus of points satisfying:

(a) $|z| = 3$

(b) $\arg(z - 1) = \frac{\pi}{4}$

(c) $|z - 2 - \mathrm{i}| = 5$

2 Sketch on Argand diagrams the regions where:

(a) $|z - 3\mathrm{i}| \leq 3$

(b) $\frac{\pi}{2} \leq \arg(z - 4 - 2\mathrm{i}) \leq \frac{5\pi}{6}$

3 Sketch on an Argand diagram the region satisfying both $|z - 1 - \mathrm{i}| \leq 3$ and $0 \leq \arg z \leq \frac{\pi}{4}$

4 (a) Sketch on an Argand diagram the circle C whose equation is $|z - \sqrt{3} - \mathrm{i}| = 1$

(b) Mark the point P on C at which $|z|$ is a minimum. Find this minimum value.

(c) Mark the point Q on C at which $\arg z$ is a maximum. Find this maximum value.

B8	Understand de Moivre's theorem and use it to find multiple angle formulae and sums of series.

Only assessed at A-level

Teaching guidance

Students should be able to:

- understand and use $(\cos\theta + \mathrm{i}\sin\theta)^n \equiv \cos(n\theta) + \mathrm{i}\sin(n\theta)$
- prove this result by induction
- find powers of complex numbers
- use the results $z + \dfrac{1}{z} = 2\cos\theta$ and $z - \dfrac{1}{z} = 2\mathrm{i}\sin\theta$ to prove trigonometric identities
- use de Moivre's theorem to find sums of series such as $\sum_{r=1}^{n}\cos(r\theta)$
- use results found to solve problems in areas such as, but not limited to, integration.

Examples

1 Show that $\cos^5\theta = \dfrac{1}{16}\left(\cos 5\theta + 5\cos 3\theta + 10\cos\theta\right)$

2 (a) Use de Moivre's theorem to show that if $z = \cos\theta + \mathrm{i}\ \sin\theta$ then $z^n + \dfrac{1}{z^n} = 2\cos n\theta$

(b) Write down a corresponding result for $z^n - \dfrac{1}{z^n}$

(c) Hence express $\cos^4\theta\ \sin^2\theta$ in the form $A\cos 6\theta + B\cos 4\theta + C\cos 2\theta + D$ where A, B, C and D are rational numbers.

3 (a) (i) Use de Moivre's Theorem to show that if $z = \cos\theta + \mathrm{i}\sin\theta$ then $z^n - \dfrac{1}{z^n} = 2\mathrm{i}\sin n\theta$

(ii) Write down a similar expression for $z^n + \dfrac{1}{z^n}$

(b) (i) Expand $\left(z - \dfrac{1}{z}\right)^2 \left(z + \dfrac{1}{z}\right)^2$ in terms of z.

(ii) Hence show that $8\sin^2\theta\cos^2\theta = A + B\cos 4\theta$ where A and B are integers.

(c) Hence, by means of the substitution $x = 2\sin\theta$, find the exact value of $\int_1^2 x^2\sqrt{4 - x^2}\,\mathrm{d}x$

4 Prove $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$

5 Prove $\displaystyle\sum_{r=1}^{n} \cos(r\theta) = \dfrac{\cos\left(\frac{1}{2}(n+1)\theta\right)\sin\left(\frac{1}{2}n\theta\right)}{\sin\left(\frac{1}{2}\theta\right)}$

B9

Know and use the definition $e^{i\theta} = \cos\theta + i\sin\theta$ and the form $z = re^{i\theta}$

Only assessed at A-level

Teaching guidance

Students should be able to:

- convert between the three forms of a complex number $z = a + bi$, $z = r(\cos\theta + i\sin\theta)$ and $z = re^{i\theta}$
- use $z = re^{i\theta}$ to simplify calculations with complex numbers
- deduce expressions for $\sin\theta$ and $\cos\theta$ in terms of exponentials.

Examples

1 Prove that $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$ and find a similar expression for $\sin\theta$

2 Express the following in the form $re^{i\theta}$

(a) $1 + i$

(b) $\sqrt{3} - i$

(c) $3 + \sqrt{3}i$

(d) $-2\sqrt{3} + 2i$

3 Verify that $z = 1 + e^{\frac{\pi i}{5}}$ is a root of the equation $(z-1)^5 = -1$

4 $z_1 = 1 + \sqrt{3}\,i$ and $z_2 = \sqrt{3} - i$

(a) Write z_1 and z_2 in the form $re^{i\theta}$ where $r > 0$ and $-\pi < \theta \le \pi$

(b) Hence show that if $z_1^5 + z_2^5 = x + iy$ then $\dfrac{y}{x} = 2 - \sqrt{3}$

5 (a) Express $e^{\frac{i\theta}{2}} - e^{-\frac{i\theta}{2}}$ in terms of $\sin\frac{\theta}{2}$

(b) Hence or otherwise, show that $\frac{1}{e^{i\theta}-1} = -\frac{1}{2} - \frac{i}{2}\cot\frac{\theta}{2}$, $\left(e^{i\theta} \neq 1\right)$

6 (a) (i) Show that $w = e^{\frac{2\pi i}{5}}$ is one of the fifth roots of unity.

(ii) Show that the other fifth roots of unity are 1, w^2, w^3 and w^4.

(b) Let $p = w + w^4$ and $q = w^2 + w^3$, where $w = e^{\frac{2\pi i}{5}}$

(i) Show that $p + q = -1$ and $pq = -1$

(ii) Write down the quadratic equations, with integer coefficients, whose roots are p and q.

(iii) Express p and q as integer multiples of $\cos\frac{2\pi}{5}$ and $\cos\frac{4\pi}{5}$, respectively.

(iv) Hence obtain the values of $\cos\frac{2\pi}{5}$ and $\cos\frac{4\pi}{5}$ in surd form.

B10	Find the n distinct nth roots of $re^{i\theta}$ for $r \neq 0$ and know that they form the vertices of a regular n-gon in the Argand diagram.

Only assessed at A-level

Teaching guidance

Students should be able to:

- find the nth roots of unity
- solve equations of the form $z^n = a + bi$ using the form $re^{i\theta}$
- understand the sum of the n distinct nth roots is zero
- show solutions on an Argand diagram.

Examples

1 Solve the following equations:

(a) $z^4 = 16i$

(b) $z^3 = 1 - i$

(c) $z^8 = 1 - \sqrt{3}i$

(d) $z^2 = -1$

(e) $(z + 1)^3 = 8i$

(f) $(z - 1)^5 = z^5$

2 (a) Write down the modulus and argument of the complex number –64.

(b) Hence solve the equation $z^4 + 64 = 0$, giving your answers in the form $r(\cos\theta + i\sin\theta)$, where $r > 0$ and $-\pi < \theta \leq \pi$

(c) Express each of these four roots in the form $a + ib$ and show, with the aid of a diagram, that the points in the complex plane that represent them form the vertices of a square.

3 (a) Express each of the complex numbers $1 + \mathrm{i}$ and $\sqrt{3} - \mathrm{i}$ in the form $r(\cos\theta + \mathrm{i}\sin\theta)$, where $r > 0$ and $-\pi < \theta \leq \pi$

(b) Using your answers to (a):

(i) show that $\dfrac{\left(\sqrt{3}-\mathrm{i}\right)^5}{\left(1+\mathrm{i}\right)^{10}} = -\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}\mathrm{i}$

giving your answer in the form $a + \mathrm{i}b$, where a and b are real numbers to be determined to two decimal places.

(ii) solve the equation $z^3 = \left(1+\mathrm{i}\right)\left(\sqrt{3}-\mathrm{i}\right)$

giving your answer in the form $a + \mathrm{i}b$, where a and b are real numbers to be determined to two decimal places.

4 (a) Show that the non-real cube roots of unity satisfy the equation $z^2 + z + 1 = 0$

(b) The real number a satisfies the equation $\dfrac{1}{a - \omega + \omega^2} + \dfrac{1}{a + \omega - \omega^2} = \dfrac{1}{2}$

where ω is one of the non-real cube roots of unity.

Find the possible values of a.

B11 Use complex roots of unity to solve geometric problems.

Only assessed at A-level

Teaching guidance

Students should be able to:

- understand and use the geometrical relationships between the nth roots of unity
- use the fact that multiplication by a root of unity is equivalent to a rotation about the origin
- use algebraic solutions to deduce geometric properties.

Examples

1 (a) Verify that $z_1 = 1 + e^{\frac{\pi i}{5}}$ is a root of the equation $(z - 1)^5 = -1$

(b) Find the other four roots of the equation.

(c) Mark on an Argand diagram the points corresponding to the five roots of the equation. Show that these roots lie on a circle and state the centre and radius of the circle.

(d) By considering the Argand diagram, find

(i) $\arg(z_1)$ in terms of π

(ii) $|z_1|$ in the form $a\cos\dfrac{\pi}{b}$, where a and b are integers to be determined.

2 (a) Write down the modulus and argument of the complex number –64.

(b) Hence solve the equation $z^4 + 64 = 0$, giving your answers in the form $r(\cos\theta + \mathrm{i}\sin\theta)$, where $r > 0$ and $-\pi < \theta \leq \pi$

(c) Express each of these four roots in the form $a + \mathrm{i}b$ and show, with the aid of a diagram, that the points in the complex plane which represent them form the vertices of a square.

(d) Using a suitable root of unity, find the complex numbers that represent the four vertices of the square in part (c) after a clockwise rotation of $\dfrac{2\pi}{3}$ radians.

3 (a) (i) Express $e^{\frac{i\theta}{2}} - e^{-\frac{i\theta}{2}}$ in terms of $\sin\frac{\theta}{2}$

(ii) Hence, or otherwise, show that $\frac{1}{e^{i\theta}-1} = -\frac{1}{2} - \frac{i}{2}\cot\frac{\theta}{2}$, $\quad (e^{i\theta} \neq 1)$

(b) Derive expressions, in the form $e^{i\theta}$ where $-\pi < \theta \leq \pi$, for the four non-real roots of the equation $z^6 = 1$.

(c) The equation $\left(\frac{w+1}{w}\right)^6 = 1 \qquad (*)$ has one real root and four non-real roots.

(i) Explain why the equation only has five roots in all.

(ii) Find the real root.

(iii) Show that the non-real roots are $\frac{1}{z_1-1}, \frac{1}{z_2-1}, \frac{1}{z_3-1}, \frac{1}{z_4-1}$

where z_1, z_2, z_3 and z_4 are the non-real roots of the equation $z^6 = 1$

(iv) Deduce that the points in an Argand diagram that represent the roots of equation $(*)$ lie on a straight line.

4 (a) Express the complex number $2 + 2i$ in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$

(b) (i) Show that one of the roots of the equation $z^3 = 2 + 2i$ is $\sqrt{2}e^{\frac{\pi i}{12}}$

(ii) Find the other two roots, giving your answers in the form $re^{i\theta}$, where r is a surd and $-\pi < \theta \leq \pi$

(c) Indicate on an Argand diagram points A, B and C corresponding to the three roots found in part (b).

(d) Find the area of the triangle ABC, giving your answer in surd form.

(e) The point P lies on the circle through A, B and C.

Denoting by w, α, β and γ the complex numbers represented by P, A, B and C, respectively, show that $\left|(w-\alpha)^2 + (w-\beta)^2 + (w-\gamma)^2\right| = 6$

C Matrices

C1 Add, subtract and multiply conformable matrices; multiply a matrix by a scalar.

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- understand what is meant by the order or dimensions of a matrix; a matrix with m rows and n columns is said to have order $m \times n$
- understand that matrices must have the same order if they are to be added or subtracted
- understand commutativity of addition; for any two matrices that can be added (ie which have the same order), $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$
- understand that two matrices can only be multiplied if the number of columns in the left matrix equals the number of rows in the right matrix
- understand non-commutativity of multiplication; it cannot be assumed that $\mathbf{AB} = \mathbf{BA}$ even when both products exist
- use the distributive law; for any matrices of appropriate orders, $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$ and $(\mathbf{U} + \mathbf{V})\mathbf{W} = \mathbf{UW} + \mathbf{VW}$
- use the associative law. For any matrices of appropriate orders, $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$
- understand how to transpose a matrix and the notation $\mathbf{M}^{\mathrm{T}}$
- understand and use the result $(\mathbf{AB})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$
- understand how to multiply (or factorise) a matrix by a scalar.

Notes

Students are expected to use a calculator to multiply matrices if possible.

Examples

1 Which of the following matrices can be multiplied by themselves?

Circle your answer(s).

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 3 & 1 \\ 1 & 2 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 3 & -1 & 1 \\ 1 & -1 & 2 \\ 2 & 1 & -3 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 1 & 1 & 4 \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

2 **A** is a 1×3 matrix and **B** is a 4×2 matrix.

Given that the products **AX**, **XB**, **BY** and **YA** can all be found, what are the orders of the matrices **X** and **Y**?

3 $\mathbf{A} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 2 & 2 \\ -1 & 3 \end{bmatrix}$

(a) Find $\mathbf{A}^2$, $\mathbf{B}^2$, **AB** and **BA**.

(b) Find **A** + **B** and verify that $(\mathbf{A} + \mathbf{B})^2 = \mathbf{A}^2 + \mathbf{AB} + \mathbf{BA} + \mathbf{B}^2$.

(c) Find $3\mathbf{A} - 2\mathbf{B}$.

4 The matrices **A** and **B** are defined by $\mathbf{A} = \begin{bmatrix} p & 2 \\ 4 & p \end{bmatrix}$ $\quad \mathbf{B} = \begin{bmatrix} 3 & 1 \\ 2 & 3 \end{bmatrix}$

(a) Find, in terms of p, the matrices:

(i) $\mathbf{A} - \mathbf{B}$

(ii) **AB**.

(b) Show that there is a value of p for which $\mathbf{A} - \mathbf{B} + \mathbf{AB} = k\mathbf{I}$, where k is an integer and $\mathbf{I}$ is the 2×2 identity matrix, and state the corresponding value of k.

5 The 2×2 matrices **A** and **B** are such that $\mathbf{AB} = \begin{bmatrix} 9 & 1 \\ 7 & 13 \end{bmatrix}$ and $\mathbf{BA} = \begin{bmatrix} 14 & 2 \\ 1 & 8 \end{bmatrix}$

Without finding **A** and **B**, determine the 2×2 matrices **C** and **D** given by $\mathbf{C} = (\mathbf{B}^T\mathbf{A}^T)$ and $\mathbf{D} = (\mathbf{A}^T\mathbf{B}^T)^T$

6 The matrices **A**, **B**, **C**, **M** and **N** are such that **M** = **AB** and **N** = **BC**

Explain why **AN** = **MC**.

C2

Understand and use zero and identity matrices.

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- understand that a zero matrix is any matrix with all its elements equal to zero, denoted by **0**
- understand and use $\mathbf{A} + \mathbf{0} = \mathbf{A}$, $\mathbf{0B} = \mathbf{0}$ and $\mathbf{C0} = \mathbf{0}$ for matrices of suitable orders
- understand that an identity matrix is any square matrix in which all of the elements on the leading diagonal are 1 and all other elements are zeros. Such a matrix is denoted by $\mathbf{I}$
- understand and use $\mathbf{AI} = \mathbf{A}$ and $\mathbf{IB} = \mathbf{B}$ for matrices of suitable orders, In particular, for any square matrix **M** of the same order as $\mathbf{I}$, $\mathbf{MI} = \mathbf{IM} = \mathbf{M}$.

Examples

1 The matrices **A** and **B** are defined in terms of a real parameter t by

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & t & 4 \\ 3 & 2 & -1 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 15 & -4 & -1 \\ -2t & 4 & 2 \\ 17 & -4 & -3 \end{bmatrix}$$

Find, in terms of t, the matrix **AB** and deduce that there exists a value of t such that **AB** is a scalar multiple of the 3×3 identity matrix $\mathbf{I}$.

2 Given that **A** is the matrix $\begin{bmatrix} -2 & 1 \\ 6 & -3 \end{bmatrix}$, find a non-zero matrix **B** such that $\mathbf{AB} = \mathbf{0}$.

3 Let the matrix $\mathbf{M} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$

(a) Show that $\mathbf{M}^2 - 3\mathbf{M} + 2\mathbf{I} = \mathbf{0}$

(b) Use the result from part (a) to prove that $\mathbf{M}^6 - 63\mathbf{M} + 62\mathbf{I} = \mathbf{0}$

C3

Use matrices to represent linear transformations in 2D; successive transformations; single transformations in 3D (3D transformations confined to reflection in one of $x = 0$, $y = 0$, $z = 0$ or rotation about one of the coordinate axes, knowledge of 3D vectors is assumed).

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- understand and use standard matrices that represent the following transformations in 2D:

Anticlockwise rotation through θ about the origin: (Given in the formulae book)	$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$
Reflection in the line $y = (\tan\theta)\,x$: (Given in the formulae book)	$\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$
Stretches parallel to the coordinate axes:	$\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix}$
Enlargement centre the origin:	$\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$

Note

Shears are not required.

- understand and use standard matrices that represent the following transformations in 3D:

  Rotations through an angle θ about one of the axes:

  (Given in the formulae book)

  $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \text{ for the } x\text{-axis}$$

  $$\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \text{ for the } y\text{-axis}$$

  $$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ for the } z\text{-axis}$$

  Reflections in simple planes

  (Not given in the formulae book)

  Reflection in $x = 0$: $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

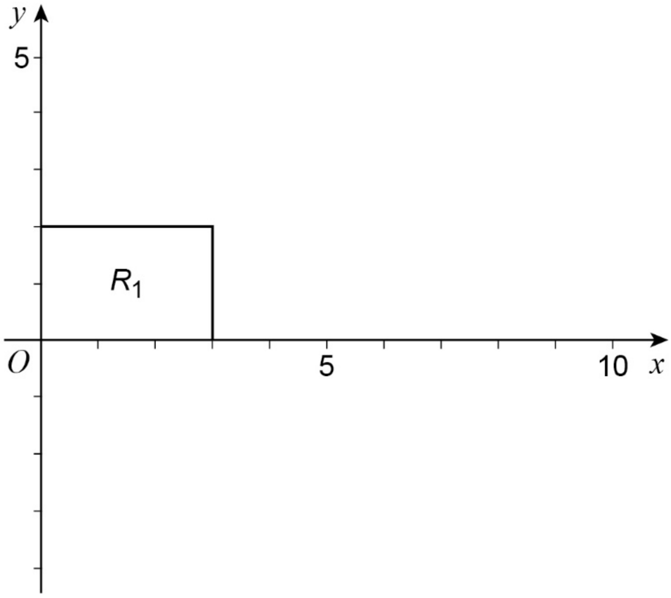
  Reflection in $y = 0$: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

  Reflection in $z = 0$: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

- use the unit square or the unit cube to deduce the matrix which represents a given transformation
- understand the order in which matrices should be multiplied when dealing with successive transformations.

Examples

1 The diagram below shows a rectangle R_1, which has vertices (0, 0), (3, 0), (3, 2) and (0, 2).



(a) On the diagram, draw

(i) the image R_2 of R_1 under a rotation through 90° clockwise about the origin

(ii) the image R_3 of R_2 under the transformation which has matrix $\begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$

(b) Find the matrix of

(i) the rotation that maps R_1 onto R_2

(ii) the combined transformation that maps R_1 onto R_3

2 (a) Write down the 2×2 matrix corresponding to each of the following transformations:

(b) a rotation about the origin through 90° clockwise

(c) a rotation about the origin through 180°.

3 The transformation T is the composition of a reflection in the line $y = x \tan \alpha$ followed by an anticlockwise rotation about O through an angle β.

Determine the matrix that represents T, and hence describe T as a single transformation.

4 The matrices $\mathbf{M}_A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$ and $\mathbf{M}_B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

represent the transformations A and B respectively.

Give a full geometrical description of each of A and B.

5 The matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -0.6 & -0.8 \\ 0 & 0.8 & -0.6 \end{bmatrix}$ represents a rotation.

(a) State the axis of rotation.

(b) Find the angle of rotation, giving your answer to the nearest degree.

C4 Find invariant points and lines for a linear transformation.

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- state invariant lines and points of some simple transformations
- understand the difference between an invariant line and a line of invariant points
- use invariant lines and points to understand and describe a transformation.

Examples

1 Find a line of invariant points for the transformation with matrix $\mathbf{M} = \begin{bmatrix} 4 & 1 & 2 \\ 2 & 6 & 6 \\ 8 & -6 & -3 \end{bmatrix}$

2 Find all invariant lines of the form $y = mx$, for the transformations given by the following matrices:

(a) $\begin{bmatrix} 5 & 15 \\ -2 & -8 \end{bmatrix}$

(b) $\begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix}$

3 The transformation T maps (x, y) to (x', y') where $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

(a) Describe the difference between an invariant line and a line of invariant points of T.

(b) Evaluate the determinant of the matrix $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

and describe the geometrical significance of the result in relation to T.

(c) Show that T has a line of invariant points and find a Cartesian equation for this line.

(d) (i) Find the image of the point $(x, -x + c)$ under T.

(ii) Hence show that all lines of the form $y = -x + c$, where c is an arbitrary constant, are invariant lines of T.

4 (a) Describe the transformation with matrix $\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$

(b) State the line of invariant points of this transformation.

C5

Calculate determinants of 2×2 matrices and 3×3 matrices and interpret as scale factors, including the effect on orientation.

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- use the notation det(**A**) or |**A**| for the determinant of a matrix **A**
- calculate determinants of 2×2 matrices, with and without a calculator
- understand and use the result det(**AB**) = det(**A**) x det(**B**).

Examples

1 The 2×2 matrices **A** and B are such that **AB** $= \begin{bmatrix} 9 & 1 \\ 7 & 13 \end{bmatrix}$ and **BA** $= \begin{bmatrix} 14 & 2 \\ 1 & 8 \end{bmatrix}$

Without finding **A** and **B**, find the value of det **B**, given that det **A** = 10.

2 Find the determinants of the following matrices:

(a) $\begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix}$

(b) $\begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 5 \\ -2 & 0 \end{bmatrix}$

Only assessed at A-level

Teaching guidance

Students should be able to:

- calculate determinants of 3×3 matrices with and without a calculator
- understand the geometrical significance of determinants of 2×2 matrices and 3×3 matrices as area and volume scale factors
- understand the implications of a negative determinant.

Examples

1 Which of the following matrices represents a transformation involving a reflection?

$$\begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix} \qquad \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix} \qquad \begin{bmatrix} 2 & 5 \\ -2 & 0 \end{bmatrix}$$

2 A 2×2 matrix **M** represents a rotation of 90º clockwise about the origin, followed by an enlargement of scale factor 2, and then a reflection in the x-axis.

(a) Write down |**M**|

(b) Check your answer to part (a) by finding **M**

3 (a) Evaluate the determinant of the matrix $\mathbf{M} = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 3 & 1 \\ 4 & 0 & -5 \end{bmatrix}$

(b) A three-dimensional shape S, with volume 12 cm^3 is transformed by a transformation with matrix **X**.

Find the volume of the image of S when

(i) $\mathbf{X} = \mathbf{M}$

(ii) $\mathbf{X} = \mathbf{MN}^2$, where **N** is a 3×3 matrix and det $\mathbf{N} = \frac{1}{3}$

C6

Understand and use singular and non-singular matrices; properties of inverse matrices.

Calculate and use the inverse of non-singular 2×2 matrices and 3×3 matrices.

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- understand and use the fact that a matrix is singular if and only if it has a determinant of zero
- understand that singular matrices do not have inverses
- understand that transformations represented by singular matrices cannot be inverted
- use the notation $\mathbf{M}^{-1}$ to denote the inverse of $\mathbf{M}$
- understand, prove and use the result $|\mathbf{M}| = \dfrac{1}{|\mathbf{M}^{-1}|}$
- understand, prove and use the result $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$
- use the definition $\mathbf{MM}^{-1} = \mathbf{M}^{-1}\mathbf{M} = \mathbf{I}$ for any non-singular matrix $\mathbf{M}$
- make use of calculator technology to find the inverse of a numeric 2×2 non-singular matrix
- understand and demonstrate the standard algorithm to find the inverse of a 2×2 non-singular matrix
- use inverse matrices to solve problems. Problems may include, for example, solving a pair of simultaneous equations, reversing a transformation or application to geometric context.

Examples

1 Which of these matrices is singular?

Circle your answer.

$$\begin{bmatrix} 2 & 5 \\ -2 & 0 \end{bmatrix} \qquad \begin{bmatrix} 2 & 5 \\ -2 & -5 \end{bmatrix} \qquad \begin{bmatrix} 0 & 5 \\ -2 & 0 \end{bmatrix} \qquad \begin{bmatrix} a & 1 \\ -1 & a \end{bmatrix}$$

2 For what values of a does the matrix $\begin{bmatrix} a & -1 \\ -1 & a \end{bmatrix}$ not have an inverse?

AQA

3 Find the inverse of the matrix $\begin{bmatrix} 3 & 4 \\ 4 & 5 \end{bmatrix}$

4 Let **A** be the matrix $\begin{bmatrix} 3 & 1 \\ 5 & -1 \end{bmatrix}$

(a) Show that $\mathbf{A}^2 - 2\mathbf{A} - 8\mathbf{I} = 0$

(b) Let the matrix $\mathbf{B} = \mathbf{A}^{-1}$.

By multiplying the equation in part (a) by **B**, find the values of the scalars α and β such that $\mathbf{B} = \alpha\mathbf{A} + \beta\mathbf{I}$

5 (a) $\mathbf{A} = \begin{bmatrix} a & 1 \\ -2 & a \end{bmatrix}$ find $\mathbf{A}^{-1}$

(b) Hence solve the simultaneous equations

$ax + y = 1$

$ay - 2x = 1$

giving your answers in terms of a.

6 It is given that **A** and **B** are matrices such that det(**AB**) = 24 and det($\mathbf{A}^{-1}$) = –3

State the value of det **B**.

7 (a) The transformation T has matrix $\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$

and maps points (x, y) onto image points (X, Y) such that $\begin{bmatrix} X \\ Y \end{bmatrix} = \mathbf{A}\begin{bmatrix} x \\ y \end{bmatrix}$

(b) Find A^{-1}.

(c) Hence express each of x and y in terms of X and Y.

Only assessed at A-level

Teaching guidance

Students should be able to:

- make use of calculator technology to find the inverse of a numeric 3 $\times$ 3 non-singular matrix
- understand and demonstrate a standard algorithm to find the inverse of a 3 $\times$ 3 non-singular matrix
- use inverse matrices to solve problems. Problems may include, for example, solving three simultaneous equations, reversing a transformation or application to geometric context.

Examples

1 For $k \neq \frac{5}{2}$, find the inverse of the matrix $\mathbf{M} = \begin{bmatrix} 1 & 3 & 1 \\ k & 0 & 1 \\ -1 & 2 & 0 \end{bmatrix}$

2 (a) The matrix **A** is defined by $\mathbf{A} = \begin{bmatrix} a & b & 0 \\ 0 & a & b \\ b & 0 & a \end{bmatrix}$

(b) Given that $\mathbf{A}^{-1}$ exists, show that $a \neq -b$

(c) Find $\mathbf{A}^{-1}$ in terms of a and b.

3 The matrix $\mathbf{P} = \begin{bmatrix} 4 & -1 & 2 \\ 1 & 1 & 3 \\ -2 & 0 & a \end{bmatrix}$, where a is a constant.

(a) (i) Find det **P** in terms of a.

(ii) Evaluate det **P** in the case when $a = 3$.

(iii) Find the value of a for which **P** is singular.

(b) The 3 $\times$ 3 matrix **Q** is such that $\mathbf{PQ} = 25\mathbf{I}$.

Without finding **Q**

(i) write down an expression for $\mathbf{P}^{-1}$ in terms of **Q**

(ii) find the value of the constant k such that $(\mathbf{PQ})^{-1} = k\mathbf{I}$

(iii) determine the numerical value of det **Q** in the case when $a = 3$.

4 (a) Find the inverse of $\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 3 & 1 \\ 2 & -1 & 1 \end{bmatrix}$

(b) The image of $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ when transformed by $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 3 & 1 \\ 2 & -1 & 1 \end{bmatrix}$ is $\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$

Find x, y and z.

C7

Solve three linear simultaneous equations in three variables by use of the inverse matrix.

Only assessed at A-level

Teaching guidance

Students should be able to:

- convert between three simultaneous linear equations and the corresponding matrix equation
- understand conditions when the equations do not have a unique solution
- find solutions by multiplying by an inverse matrix.

Examples

1 Show that the system of equations

$$3x - y + 3z = 11$$
$$4x + y - 5z = 17$$
$$5x - 4y + 14z = 16$$

does not have a unique solution.

2 The matrix $\mathbf{A}$ is given by $\mathbf{A} = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

(a) Given that $\mathbf{A}^{-1} = \frac{1}{6}\begin{bmatrix} r & -2 & 2 \\ -1 & 5 & -2 \\ -2 & s & 2 \end{bmatrix}$ find the value of r and the value of s.

(b) Hence, or otherwise, find the solution of the system of equations

$$x - z = k$$
$$x + 2y + z = 5$$
$$2x + 2y + 3z = 7$$

giving your answers in terms of k.

3 The matrix $\mathbf{M} = \begin{bmatrix} 1 & 4 & 2 \\ 3 & k & 3 \\ 2 & k & 1 \end{bmatrix}$, where k is a constant.

(a) Show that **M** is non-singular for all values of k.

(b) Obtain $\mathbf{M}^{-1}$ in terms of k.

(c) Use $\mathbf{M}^{-1}$ to solve the equations

$$x + 4y + 2z = 25$$
$$3x + ky + 3z = 3$$
$$2x + ky + z = 2$$

giving your solution in terms of k.

C8

Interpret geometrically the solution and failure of solution of three simultaneous linear equations.

Only assessed at A-level

Teaching guidance

Students should be able to:

- understand that equations in three variables represent planes
- distinguish between:

The equations have no solutions, they are said to be inconsistent.	Two of the planes are distinct and parallel. or The three planes form a triangular prism.
There are infinitely many solutions, but no unique solution. The equations are consistent.	The three planes form a sheaf, and intersect along a line, or two of the planes are the same.
There is a unique solution, the equations are consistent.	The three planes intersect at a unique point.

Examples

1 A system of equations is given by

$$x + 3y + 5z = -2$$
$$3x - 4y + 2z = 7$$
$$ax + 11y + 13z = b$$

where a and b are constants.

(a) Find the unique solution of the system in the case when $a = 3$ and $b = 2$

(b) (i) Determine the value of a for which the system does not have a unique solution.

(ii) For this value of a, find the value of b such that the system of equations is consistent.

2 Consider the following system of equations, where k is a real constant.

$$kx + 2y + z = 5$$
$$x + (k - 1)y + 2z = 3$$
$$2x + 2ky + 3z = -11$$

(a) Show that the system does not have a unique solution when $k^2 = 16$

(b) In the case when $k = 4$, show that the system is inconsistent.

(c) In the case when $k = -4$

(i) solve the system of equations

(ii) interpret this result geometrically.

C9 Factorisation of determinants using row and column operations.

Only assessed at A-level

Teaching guidance

Students should be able to:

- understand and use permissible row and column operations to find factors
- use symmetry to find factors
- use the factor theorem to find factors.

Examples

1 Factorise completely the determinant $\begin{vmatrix} y & x & x+y-1 \\ x & y & 1 \\ y+1 & x+1 & 2 \end{vmatrix}$

2 (a) Show that $(a-b)$ is a factor of the determinant $\Delta = \begin{vmatrix} a & b & c \\ b+c & c+a & a+b \\ bc & ca & ab \end{vmatrix}$

(b) Factorise Δ completely into linear factors.

3 Express the determinant $\begin{vmatrix} 1 & a^2 & bc \\ 1 & b^2 & ca \\ 1 & c^2 & ab \end{vmatrix}$ as the product of four linear factors.

C10	Find eigenvalues and eigenvectors of 2×2 and 3×3 matrices. Find and use the characteristic equation. Understand the geometrical significance of eigenvalues and eigenvectors.

Only assessed at A-level

Teaching guidance

Students should be able to:

- understand the definitions of eigenvalues and eigenvectors
- form and solve the characteristic equation to find eigenvalues
- find eigenvectors for corresponding eigenvalues
- understand and use links to invariance.

Examples

1 Find the eigenvalues and eigenvectors of the matrix $\mathbf{M} = \begin{bmatrix} -8 & -4 \\ 25 & 12 \end{bmatrix}$

2 Let $\mathbf{Y} = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$

(a) Show that 4 is a repeated eigenvalue of **Y**, and find the other eigenvalue of **Y**.

(b) For each eigenvalue of **Y**, find a full set of eigenvectors.

(c) The matrix **Y** represents the transformation **T**.
Describe the geometrical significance of the eigenvectors of **Y** in relation to **T**.

3 The non-singular matrix $\mathbf{M} = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix}$

(a) (i) Show that $\mathbf{M}^2 + 2\mathbf{I} = k\mathbf{M}$ for some integer k to be determined.

(ii) By multiplying the equation in part (i) by $\mathbf{M}^{-1}$, show that $\mathbf{M}^{-1} = a\mathbf{M} + b\mathbf{I}$ for constants a and b to be found.

(b) (i) Determine the characteristic equation of **M** and show that **M** has a repeated eigenvalue, 1, and another eigenvalue, 2.

(ii) Give a full set of eigenvectors for each of these eigenvalues.

(iii) State the geometrical significance of each set of eigenvectors in relation to the transformation with matrix **M**.

4 (a) Given that –1 is an eigenvalue of the matrix $\mathbf{M} = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{bmatrix}$

find a corresponding eigenvector.

(b) Determine the other two eigenvalues of **M**, expressing each answer in its simplest surd form.

5 The matrix $\mathbf{M} = \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}$ represents a transformation.

(a) Find det **M** and give a geometrical interpretation of this result.

(b) Show that the characteristic equation of **M** is $\lambda^2 - 2\lambda + 1 = 0$, where is an eigenvalue of **M**.

(c) Hence find an eigenvector of **M**.

(d) Write down the equation of the line of invariant points of the transformation.

C11 Diagonalisation of matrices; $\mathbf{M} = \mathbf{UDU}^{-1}$; $\mathbf{M}^n = \mathbf{UD}^n\mathbf{U}^{-1}$ when eigenvalues are real.

Only assessed at A-level

Teaching guidance

Students should be able to:

- understand the term diagonal matrix
- understand how to use the eigenvalues and eigenvectors to form **U** and **D** in $\mathbf{M} = \mathbf{UDU}^{-1}$
- understand when it is not possible to write **M** as $\mathbf{M} = \mathbf{UDU}^{-1}$
- use $\mathbf{M} = \mathbf{UDU}^{-1}$ to find $\mathbf{M}^n$ and show how terms cancel.

Examples

1 Express $\begin{bmatrix} 4 & 2 \\ -3 & -1 \end{bmatrix}$ in the form $\mathbf{VDV}^{-1}$, where **D** is a diagonal matrix.

2 The 2×2 matrix **M** has an eigenvalue 3, with corresponding eigenvector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

and a second eigenvalue –3, with corresponding eigenvector $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

The diagonalised form of **M** is $\mathbf{M} = \mathbf{UDU}^{-1}$

(a) (i) Write down suitable matrices **D** and **U** and find $\mathbf{U}^{-1}$.

(ii) Hence determine the matrix **M**

(b) Given that n is a positive integer, use the result $\mathbf{M}^n = \mathbf{U}\,\mathbf{D}^n\mathbf{U}^{-1}$ to show that:

(i) when n is even, $\mathbf{M}^n = 3^n\mathbf{I}$

(ii) when n is odd, $\mathbf{M}^n = 3^{n-1}\mathbf{M}$

3 For all real numbers a and b, with $b \neq 0$ and $b \neq \pm a$, the matrix $\mathbf{M} = \begin{bmatrix} a & b+a \\ b-a & -a \end{bmatrix}$

(a) (i) Show that the eigenvalues of **M** are b and $-b$

(ii) Show that $\begin{bmatrix} b+a \\ b-a \end{bmatrix}$ is an eigenvector of **M** with eigenvalue b

(iii) Find an eigenvector of **M** corresponding to the eigenvalue $-b$

(b) By writing **M** in the form $\mathbf{UDU}^{-1}$, for some suitably chosen diagonal matrix **D** and corresponding matrix **U**, show that $n\mathbf{M}^{11} = b^{10}\mathbf{M}$

4 (a) Find the eigenvalues and corresponding eigenvectors of the matrix $\mathbf{X} = \begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix}$

(b) (i) Write down a diagonal matrix **D** and a suitable matrix **U**, such that $\mathbf{X} = \mathbf{UDU}^{-1}$

(ii) Write down the matrix $\mathbf{U}^{-1}$

(iii) Use your results from parts (i) and (ii) to determine the matrix $\mathbf{X}^5$

5 (a) Express $\mathbf{M} = \frac{1}{6}\begin{bmatrix} -1 & 9 & -5 \\ 1 & 3 & -1 \\ -4 & 0 & -2 \end{bmatrix}$ in the form $\mathbf{VDV}^{-1}$, where **D** is a diagonal matrix.

(b) Hence find $\mathbf{M}^n$ for any integer n.

D Further algebra and functions

D1 Understand and use the relationship between roots and coefficients of polynomial equations up to quartic equations.

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- understand and use the sum and product of the roots of a quadratic equation:

  For $ax^2 + bx + c = 0$ with roots α and β then $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$

- understand and use the relationship between the roots of a cubic equation and its coefficients:

  For $ax^3 + bx^2 + cx + d = 0$ with roots α, β and γ then $\sum \alpha = \alpha + \beta + \gamma = -\frac{b}{a}$,

  $\sum \alpha\beta = \alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$ and $\alpha\beta\gamma = -\frac{d}{a}$

- understand and use the relationship between the roots of a quartic equation and its coefficients:

  For $ax^4 + bx^3 + cx^2 + dx + e = 0$ with roots α, β, γ and δ then $\sum \alpha = -\frac{b}{a}$, $\sum \alpha\beta = \frac{c}{a}$,

  $\sum \alpha\beta\gamma = -\frac{d}{a}$ and $\alpha\beta\gamma\delta = \frac{e}{a}$

- use the relationships between the roots to form new equations
- recall or derive and use standard algebraic results such as, but not limited to:

  $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

  $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$

  $\sum \alpha^2 = \left(\sum \alpha\right)^2 - 2\sum \alpha\beta$

Examples

1 The quadratic equation $2x^2 + 8x + 1 = 0$ has roots α and β

(a) Write down the value of $\alpha + \beta$ and the value of $\alpha\beta$

(b) (i) Find the value of $\alpha^2 + \beta^2$

(ii) Hence or otherwise, show that $\alpha^4 + \beta^4 = \dfrac{449}{2}$

(c) Find a quadratic equation, with integer coefficients, which has roots

$$2\alpha^4 + \frac{1}{\beta^2} \text{ and } 2\beta^4 + \frac{1}{\alpha^2}$$

2 The cubic equation $27z^3 + kz^2 + 4 = 0$ has roots α, β and γ

(a) Write down the values of $\alpha\beta + \beta\gamma + \gamma\alpha$ and $\alpha\beta\gamma$

(b) (i) In the case where $\beta = \gamma$, find the roots of the equation.

(ii) Find the value of k in this case.

(c) (i) In the case where $\alpha = 1 - \mathrm{i}$, find α^2 and α^3

(ii) Hence find the value of k in this case.

(d) In the case where $k = -12$, find a cubic equation with integer coefficients which has roots $\dfrac{1}{\alpha} + 1$, $\dfrac{1}{\beta} + 1$ and $\dfrac{1}{\gamma} + 1$

3 The roots of the equation $x^4 - px^3 + qx^2 - pqx + 1 = 0$ are α, β, γ, and δ.

Show that $(\alpha + \beta + \gamma)(\alpha + \beta + \delta)(\alpha + \gamma + \delta)(\beta + \gamma + \delta) = 1$

4 The equation $x^3 + 7x + 8 = 0$ has roots α, β and γ.

Find the equation whose roots are $\alpha^2 + 1$, $\beta^2 + 1$ and $\gamma^2 + 1$.

D2

Form a polynomial equation whose roots are a linear transformation of the roots of a given polynomial equation (of at least cubic degree).

Assessed at AS and A-level

Teaching guidance

Students should be able to apply a linear transformation of the form $X \to aX + b$ to form an equation with the required roots.

Examples

1 The cubic equation $x^3 - 3x^2 + 4 = 0$ has roots α, β and γ

Find the cubic equations with roots

(a) 2α, 2β and 2γ

(b) $\alpha - 2$, $\beta - 2$ and $\gamma - 2$

2 The roots of the equation $x^4 - px^3 + x^2 - 6x + 1 = 0$ are α, β, γ, and δ.

Find the quartic equation whose roots are $\frac{\alpha}{2}, \frac{\beta}{2}, \frac{\gamma}{2}$ and $\frac{\delta}{2}$

D3 Understand and use formulae for the sums of integers, squares and cubes and use these to sum other series.

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- understand and apply the formulae:
  $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$, $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$ and $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$
- prove these results.

Notes

All three formulae are given in the formula booklet, the first of them in a more general form as the sum of an arithmetic series.

Examples

1 Show that $\sum_{r=1}^{n}\left(r^2 - r\right) = kn\left(n+1\right)\left(n-1\right)$ where k is a rational number.

2 (a) Use the formulae for $\sum_{r=1}^{n} r^2$ and $\sum_{r=1}^{n} r$ to show that $\sum_{r=1}^{n} 2r\left(3r+2\right) = n\left(n+p\right)\left(2n+q\right)$

where p and q are integers.

(b) (i) Express $\log_8 4^r$ in the form λr, where λ is a rational number.

(ii) By first finding a suitable cubic inequality for k, find the greatest value of k for which

$$\sum_{r=k+1}^{60}\left(3r+2\right)\log_8 4^r \text{ is greater than } 106\ 060$$

3 Find an expression in terms of n for $\sum_{r=n}^{2n}(r+1)^2$

4 (a) Find $\sum_{r=1}^{n}\left(r^3 - 6r\right)$ expressing your answer in the form $kn(n + 1)(n + p)(n + q)$

where k is rational and p and q are integers.

(b) It is given that $S = \sum_{r=1}^{1000}\left(r^3 - 6r\right)$

Without calculating the value of S, show that S is a multiple of 2008.

5 (a) (i) Expand $(2r - 1)^2$

(ii) Hence show that $\sum_{r=1}^{n}(2r - 1)^2 = \frac{1}{3}n\left(4n^2 - 1\right)$

(b) Hence find the sum of the squares of the odd numbers between 100 and 200

D4

Understand and use the method of differences for summation of series including use of partial fractions.

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- demonstrate how to use the method of differences to find the summation of a series, clearly showing how the process works
- select and use the method of differences, recognising when the method can be applied appropriately.

Examples

1 Use the method of differences to find an expression for $\sum_{r=1}^{n} \frac{1}{r} - \frac{1}{r+1}$

2 (a) Simplify $r^2\left(r+1\right)^2 - (r-1)^2 r^2$

(b) Hence prove the result $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$

Only assessed at A-level

Teaching guidance

Students should be able to rearrange an expression using partial fractions so that the method of differences may be applied.

Example

1 Sum the series $\sum_{r=1}^{n} \frac{1}{r\left(r+1\right)\left(r+2\right)\left(r+3\right)}$

D5

Find the Maclaurin series of a function including the general term.

Only assessed at A-level

Teaching guidance

Students should be able to:

- understand and use $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + ... + \frac{x^r}{r!}f^{(r)}(0) + ...$ to find the series expansion of a function
- use Maclaurin series to solve problems, which may include finding approximations for particular values of a function or definite integrals.

Notes

Standard Maclaurin series and the general form are given in the formulae booklet.

Examples

1 Prove that the first two non-zero terms in the Maclaurin series expansion, in ascending powers of x, of tan x are $x + \frac{x^3}{3}$.

2 (a) Use Maclaurin's series to find an expression for $\sin^{-1}x$ as a series of ascending powers of x up to and including x^3

(b) Show how you can use the expansion you found in part (a) to find an approximation for π

3 (a) Find the Maclaurin expansion for $\ln\cos x$ in ascending powers of x up to and including x^4

(b) By substituting an appropriate value for x show that $\ln 2 \approx \frac{\pi^2}{16}\left(1 + \frac{\pi^2}{96}\right)$

D6

Recognise and use the Maclaurin series for e^x, $\ln(1 + x)$, $\sin x$, $\cos x$, and $(1 + x)^n$, and be aware of the range of values of x for which they are valid (proof not required).

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- understand and apply the standard results

$$\mathrm{e}^x = \exp(x) = 1 + x + \frac{x^2}{2!} + \ldots \frac{x^r}{r!} + \ldots \text{ for all } x$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{r+1}\frac{x^r}{r} + \ldots \qquad (-1 < x \le 1)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^r \frac{x^{2r+1}}{(2r+1)!} + \ldots \text{ for all } x$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^r \frac{x^{2r}}{(2r!)} + \ldots \text{ for all } x$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{1.2\ldots r}x^r + \ldots \qquad (|x| < 1, n \in \mathbb{Q})$$

- use these standard results to find expansions of related functions and their range of validity.

Notes

These expansions and ranges of validity are all given in the formulae book.

1 (a) Expand each of the following functions as a series in ascending powers of x, up to and including the term in x^3

(i) $\ln(1 - x)$

(ii) $\mathrm{e}^x(1 - 2x)^3$

(iii) $\mathrm{e}^{-2x} + 2 \sin x$

(iv) $\dfrac{\ln(1+x)}{1+3x}$

(b) For each of the series expansions, determine the range of values for x for which the expansion is valid.

2 (a) Expand $\ln(1 - 2x)$ as a series in ascending powers of x, up to and including the term in x^3

(b) Determine the range of validity of this series.

3 The function $f(x)$ is defined by $f\left(x\right)=a(1+2x)^{\frac{1}{2}}-\ln\left(1+3x\right)$ where a is a contant.

When $f(x)$ is expanded as a series in ascending powers of x, there is no term in x

(a) Find the value of a

(b) Obtain the first two non-zero terms in the expansion.

(c) Determine the values of x for which the expansion of $f(x)$ is valid.

D7 Evaluation of limits using Maclaurin series or l'Hôpital's rule.

Only assessed at A-level

Teaching guidance

Students should be able to:

- demonstrate an understanding of the limiting process used
- use Maclaurin's series together with any necessary algebraic manipulation before taking limits
- understand the circumstances under which l'Hôpital's Rule may be applied:
  $\lim_{x\to c}\frac{\mathrm{f}(x)}{\mathrm{g}(x)}=\lim_{x\to c}\frac{\mathrm{f}'(x)}{\mathrm{g}'(x)}$, provided that $\mathrm{f}(x)=\mathrm{g}(x)=0$ or $\mathrm{f}(x)=\mathrm{g}(x)=\pm\infty$ and $\lim_{x\to c}\frac{\mathrm{f}'(x)}{\mathrm{g}'(x)}$ exists
- apply l'Hôpital's rule once, or more than once, together with any necessary algebraic manipulation to find a limit.

Examples

1 (a) Show that $\ln(1+\sin x)=x-\frac{1}{2}x^2+\frac{1}{6}x^3+\ldots$

(b) Hence find $\lim_{x\to 0}\frac{\ln(1+\sin x)-x}{x^2}$

2 (a) Find the first three non-zero terms in the expansion of $\frac{x}{\ln(1+x)}$

as a series in ascending powers of x

(b) Hence find $\lim_{x\to 0}\left(\frac{1}{\ln(1+x)}-\frac{1}{x}\right)$

3 Using l'Hôpital's rule evaluate the limit $\lim_{x\to 0}\frac{x-\sin x}{x^3}$

D8

Inequalities involving polynomial equations (cubic and quartic).

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- understand how to manipulate inequalities correctly (avoiding errors introduced through multiplication by negatives etc)
- consider critical values and give solution sets using correct interval or set notation
- use a calculator find numerical solutions.

Examples

1 Solve the inequality $(x+1)(x+4)(x-1)(x-2) < 2(x-1)^2(x-2)^2$

2 The function f is defined by $\mathrm{f}(x) = x^4 - 6x^3 + 4x^2 + 6x - 5$, $x \in \mathbb{R}$

(a) Solve the inequality $\mathrm{f}(x) < 0$

(b) $\mathrm{f}(x)$ is transformed by a stretch in the y-direction scale factor $-a$, where a is a positive constant, to form the function $\mathrm{g}(x)$

Solve the inequality $\mathrm{g}(x) < 0$

D9

Solving inequalities such as $\dfrac{ax+b}{cx+d} < ex+f$ algebraically.

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- solve such inequalities by multiplying by the square of the denominator and understand why this is important
- solve such inequalities by rearranging to the form $\dfrac{ax+b}{cx+d} - \dfrac{(ex+f)(cx+d)}{cx+d} < 0$ and considering critical values.

Examples

1 Solve the inequality $\dfrac{6x}{x-1} < 3$

2 A student solves the inequality $\dfrac{9-3x}{x-3} < 7$

The student's working is shown below:

$$\frac{9-3x}{x-3} < 7$$

$$\Rightarrow 9-3x < 7(x-3)$$

$$\Rightarrow 9-3x < 7x-21$$

$$\Rightarrow 30 < 10x$$

$$\therefore x > 3$$

Explain the error in the student's reasoning.

AQA

D10 Modulus of functions and associated inequalities.

Only assessed at A-level

Teaching guidance

Students should be able to:

- understand how to manipulate expressions containing modulus functions
- solve equations and inequalities involving modulus functions.

Examples

1 Solve the inequality $\left|\dfrac{2x-1}{x+5}\right| > 1$

2 (a) Solve the equation $\left|x^2-4\right| = 2x+4$

(b) Hence solve $\left|x^2-4\right| > 2x+4$

3 Find the exact solutions of the equation $\left|\sinh x\right| = 1$

D11

Graphs of $y = \left|\mathrm{f}(x)\right|$, $y = \dfrac{1}{\mathrm{f}(x)}$ for given $y = \mathrm{f}(x)$.

Only assessed at A-level

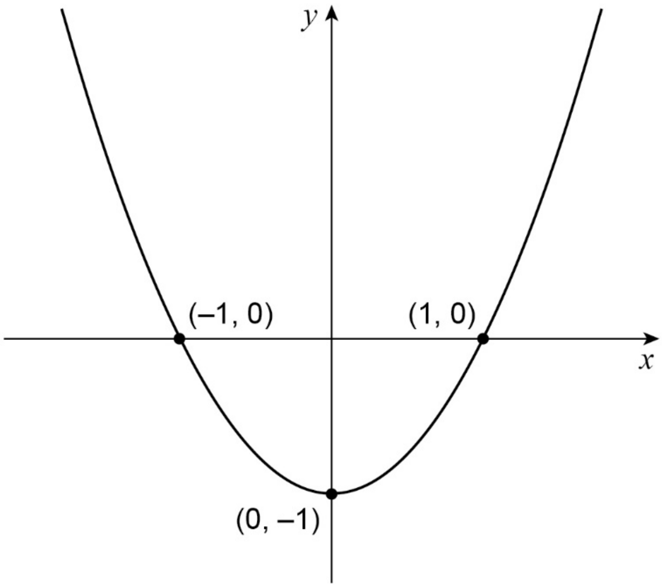
Teaching guidance

Students should be able to:

- produce clear sketches with $\mathrm{f}(x)$ given either algebraically or in graphical form
- include important features on their sketches such as intercepts with coordinate axes, stationary points and asymptotes
- analyse the behaviour of a graph near critical values
- understand the behaviour of the reciprocal of $\mathrm{f}(x)$ as $\mathrm{f}(x) \to 0$ or $\mathrm{f}(x) \to \pm\infty$
- use their sketches to solve problems.

Examples

1 The graph of $y = f(x)$ is shown below:



(a) Sketch the graph of $y = |f(x)|$

(b) Sketch the graph of $y = \dfrac{1}{f(x)}$

2 By drawing a suitable sketch, show that the equation $|\sinh x| + \cosh x = 2$ has exactly two solutions.

D12

Graphs of rational functions of form $\dfrac{ax+b}{cx+d}$; asymptotes, points of intersection with coordinate axes or other straight lines; associated inequalities.

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- produce clear and useful sketches
- include important features on their sketches such as intercepts with coordinate axes and asymptotes
- solve simultaneous equations to find points of intersection
- use their sketches to solve inequalities.

Examples

1 A curve has equation $y = \dfrac{3x-1}{x+2}$

(a) Write down the equations of the two asymptotes to the curve.

(b) Sketch the curve, indicating the coordinates of the points where the curve intersects the coordinate axes.

(c) Hence, or otherwise, solve the inequality $0 < \dfrac{3x-1}{x+2} < 3$

2 (a) Sketch the graphs of $y = \dfrac{x+1}{x+2}$ and $2y - 2x = 1$ on the same axes, clearly labelling any points of intersection and asymptotes.

(b) Hence, solve the inequality $\dfrac{2x+2}{x+2} \geq 2x+1$

AQA

D13

Graphs of rational functions of form $\dfrac{ax^2+bx+c}{dx^2+ex+f}$, including cases when some of these coefficients are zero; asymptotes parallel to coordinate axes; oblique asymptotes.

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- draw clear sketches
- include important features on their sketches such as intercepts with coordinate axes and asymptotes parallel to the coordinate axes
- solve simultaneous equations to find points of intersection
- use their sketches to solve problems.

Examples

1 A curve has equation $y=\dfrac{1}{x^2-4}$

(a) (i) Write down the equations of the three asymptotes of the curve.

(ii) Sketch the curve, showing the coordinates of any points of intersection with the coordinate axes.

(b) Hence, or otherwise, solve the inequality $\dfrac{1}{x^2-4}<-2$

2 (a) Find the equations of the asymptotes of the curve $y=\dfrac{x^2+2x}{x^2-2x+1}$

(b) Given that the curve has one stationary point, sketch the graph of $y=\dfrac{x^2+2x}{x^2-2x+1}$

Only assessed at A-level

Teaching guidance

Students should be able to:

- understand and use the concept of an oblique asymptote
- find the equation of an oblique asymptote.

Examples

1 Find the equations of the asymptotes of the curve $y = \dfrac{x^2 - x + 1}{x + 1}$

2 A curve, with equation $y = \dfrac{ax^2 + bx + 1}{2x + 1}$, has an asymptote $y = 2x$

(a) Find the values of a and b

(b) Write down the equation of the other asymptote.

(c) Without using calculus, find the coordinates of the turning points.

(d) Sketch the curve.

D14

Using quadratic theory (not calculus) to find the possible values of the function and coordinates of the stationary points of the graph for rational functions of form $\dfrac{ax^2+bx+c}{dx^2+ex+f}$

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- use the discriminant of a quadratic to determine when a horizontal line is a tangent to a curve and deduce the turning points of the curve
- use the discriminant of a quadratic to determine when a horizontal line does (or does not) intersect a curve and deduce the range of the associated function.

Note

The term ‘range’ will not be used at AS.

Examples

1 A graph has equation $y=\dfrac{x-4}{x^2+9}$

(a) Explain why the graph has no vertical asymptote and give the equation of the horizontal asymptote.

(b) Show that, if the line $y=k$ intersects the graph, the x-coordinates of the points of intersection of the line with the graph must satisfy the equation $kx^2-x+(9k+4)=0$

(c) Show that this equation has real roots if $-\dfrac{1}{2}\le k\le\dfrac{1}{18}$

(d) Hence find the coordinates of the two stationary points on the curve.

Note: no credit will be given for methods involving differentiation.

2 A curve C has the equation $y = \dfrac{2}{x(x-4)}$

(a) Write down the equations of the three asymptotes of C.

(b) The curve C has one stationary point.

By considering an appropriate quadratic equation, find the coordinates of this stationary point.

Note: no credit will be given for solutions based on differentiation.

(c) Sketch the curve C.

Only assessed at A-level

Teaching guidance

Students should understand the language associated with functions. Overarching theme 1.4 “Understand and use the definition of a function; domain and range of functions.” is not assessed in AS Maths and Further Maths so the vocabulary of functions can only be introduced in A-level questions.

Examples

1 The function f is defined by $\mathrm{f}(x) = \dfrac{4x^2 - x - 3}{2x^2 - x - 3}$ with domain $\{x \in \mathbb{R} : x \neq a, x \neq b\}$

Clearly showing your reasoning,

(a) find the values of a and b

(b) find the equations of all the asymptotes to the curve $y = \mathrm{f}(x)$

(c) without using calculus, find the range of f

(d) find the coordinates of the two turning points.

(e) sketch the curve.

D15

Sketching graphs of curves with equations $y^2 = 4ax$, $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, $xy = c^2$ including intercepts with axes and equations of asymptotes of hyperbolas.

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- recognise the equations of a parabola, hyperbola and ellipse and sketch their graphs
- find points of intersection with the coordinate axes, straight lines or other curves
- find the equations of asymptotes, tangent and normal.

Notes

Equations of the asymptotes of the general hyperbola are given in the formulae book.

Examples

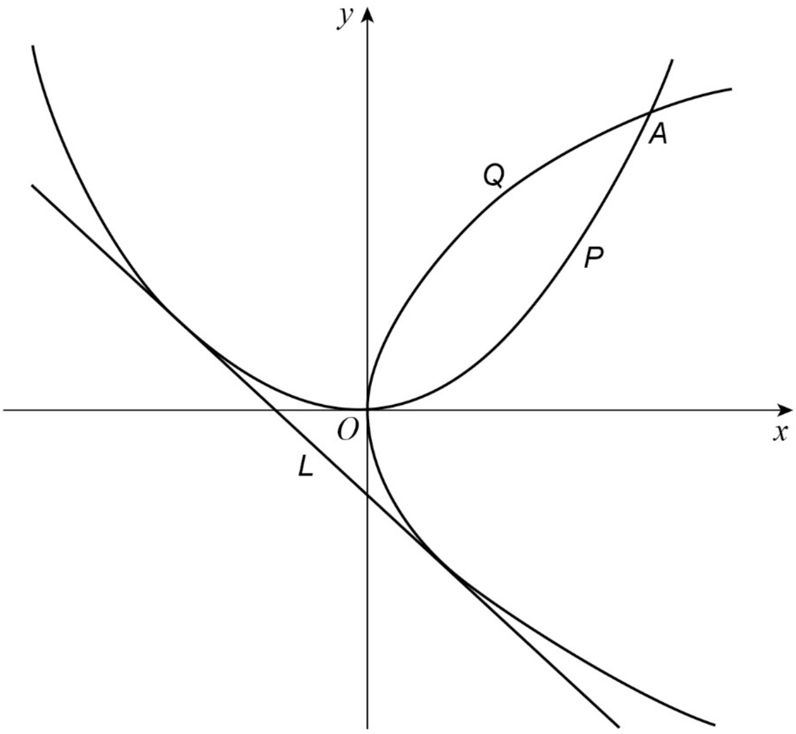
1 A curve C_1 has equation $\dfrac{x^2}{9} - \dfrac{y^2}{16} = 1$

Sketch the curve C_1, stating the values of its intercepts with the coordinate axes.

2 The diagram shows a parabola, *P*, which has equation $y = \frac{1}{8}x^2$, and another parabola, *Q*, which is the image of *P* under a reflection in the line $y = x$

The parabolas *P* and *Q* intersect at the origin and again at a point *A.*

The line *L* is a tangent to both *P* and *Q*.



(a) (i) Find the coordinates of the point *A*.

(ii) Write down an equation for *Q*.

(iii) Give a reason why the gradient of *L* must be –1

(b) (i) Given that the line $y = -x + c$ intersects the parabola *P* at two distinct points, show that $c > -2$

(ii) Find the coordinates of the points at which the line *L* touches the parabolas *P* and *Q*.

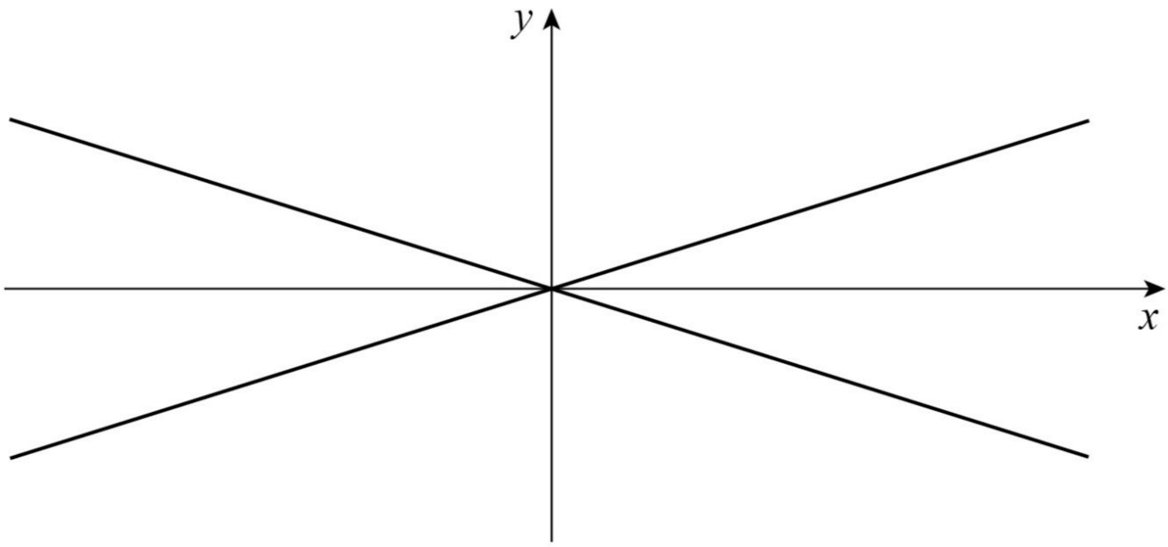
Note: no credit will be given for solutions based on differentiation.

3 A hyperbola *H* has the equation $\frac{x^2}{9} - y^2 = 1$

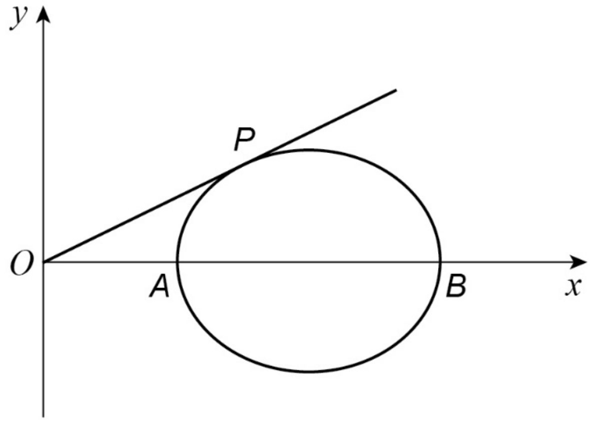
(a) Find the equations of the asymptotes of *H.*

(b) The asymptotes of *H* are shown in the diagram below.

On the same diagram, sketch the hyperbola *H*. Indicate on your sketch the coordinates of the points of intersection of *H* with the coordinate axes.



4 An ellipse is shown below.



The ellipse intersects the x-axis at the points A and B.

The equation of the ellipse is $\frac{(x-4)^2}{4} + y^2 = 1$

(a) Find the x-coordinates of A and B.

(b) The line $y = mx$ $(m > 0)$ is a tangent to the ellipse, with point of contact P.

(i) Show that the x-coordinate of P satisfies the equation $(1 + 4m^2)x^2 - 8x + 12 = 0$

(ii) Hence find the exact value of m

(iii) Find the coordinates of P.

D16

Single transformations of curves involving translations, stretches parallel to coordinate axes and reflections in the coordinate axes and the lines $y = \pm x$.

Extend to composite transformations including rotations and enlargements.

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- determine the equation of a parabola, hyperbola or ellipse following a single transformation of either a stretch parallel to one of the axes, a reflection in one of the axes, a reflection in $y = \pm x$ or a translation
- determine the transformation that has been applied by considering the equation.

Examples

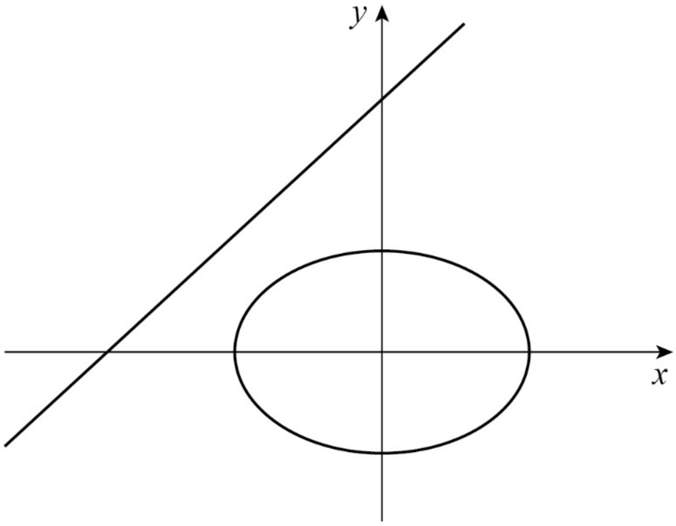
1 A curve C_1 has equation $\dfrac{x^2}{9} - \dfrac{y^2}{16} = 1$

(a) Sketch the curve C_1, stating the values of its intercepts with the coordinates axes.

(b) The curve C_1 is translated by the vector $\begin{bmatrix} k \\ 0 \end{bmatrix}$, where $k < 0$, to give a curve C_2

Given that C_2 passes through the origin, find the equations of the asymptotes of C_2

2 The diagram shows the ellipse E with equation $\frac{x^2}{5} + \frac{y^2}{4} = 1$ and the straight line L with equation $y = x + 4$



(a) Write down the coordinates of the points where the ellipse E intersects the coordinate axes.

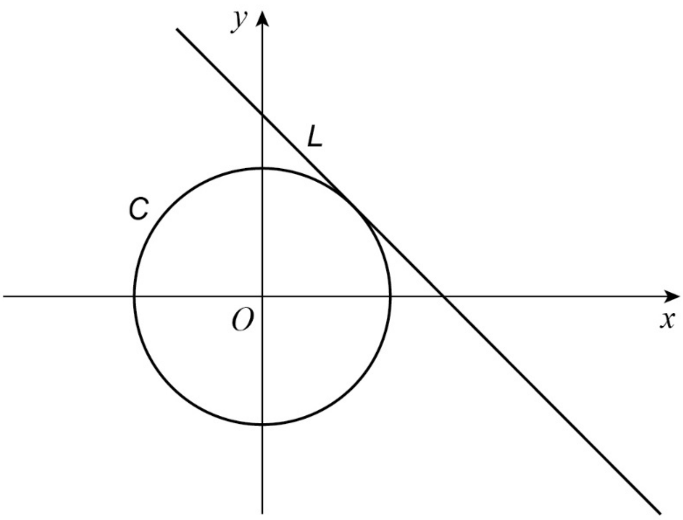
(b) The ellipse E is translated by the vector $\begin{bmatrix} p \\ 0 \end{bmatrix}$ where p is a constant.

Write down the equation of the translated ellipse.

(c) Show that, if the translated ellipse intersects the line L, the x-coordinates of the points of intersection must satisfy the equation $9x^2 - (8p - 40)x + (4p^2 + 60) = 0$

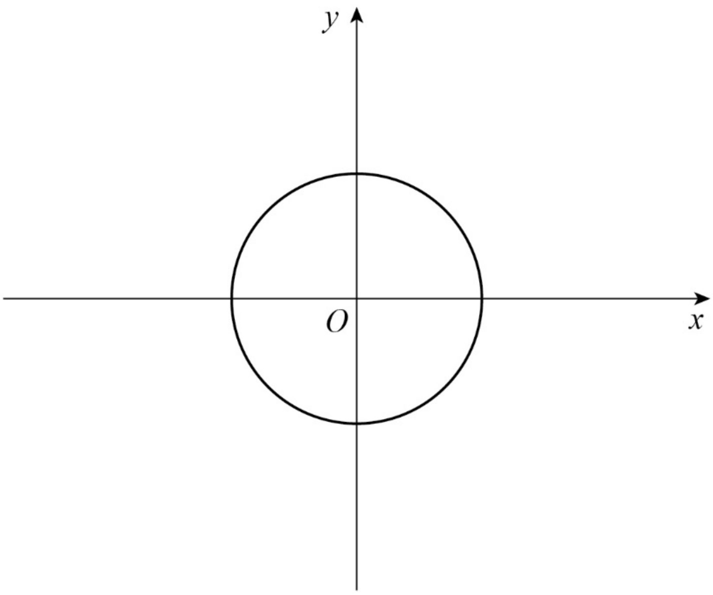
(d) Given that the line L is a tangent to the translated ellipse, find the coordinates of the two possible points of contact.

3 The diagram shows a circle C and a line L, which is the tangent to C at the point (1, 1). The equations of C and L are $x^2 + y^2 = 2$ and $x + y = 2$ respectively.



The circle C is now transformed by a stretch with scale factor 2 parallel to the x-axis. The image C under this stretch is an ellipse E.

(a) On the diagram below, sketch the ellipse E, indicating the coordinates of the points where it intersects the coordinate axes.

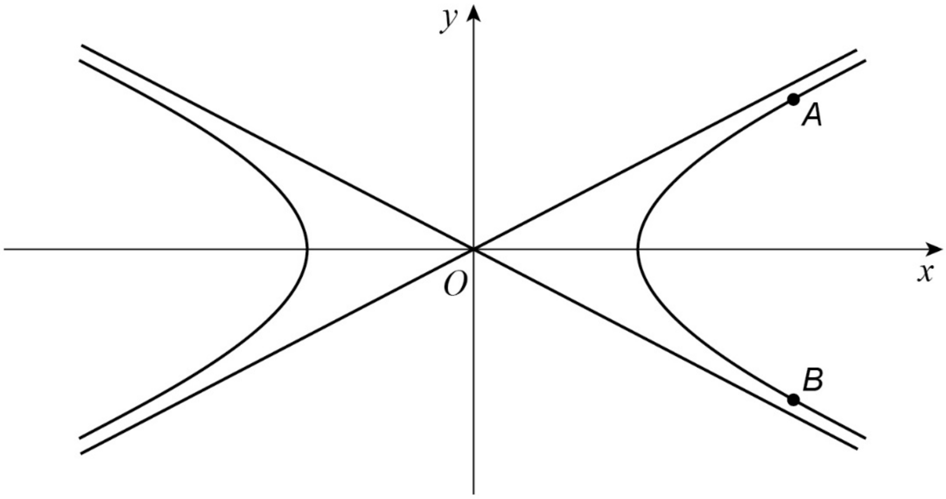


(b) Find the equations of:

(i) the ellipse E

(ii) the tangent to E at the point (2, 1)

4 The diagram shows the hyperbola $\frac{x^2}{4} - y^2 = 1$ and its asymptotes.



(a) Write down the equations of the two asymptotes.

(b) The points on the hyperbola for which $x = 4$ are denoted by *A* and *B*.

Find, in surd form, the y-coordinates of *A* and *B*.

(c) The hyperbola and its asymptotes are translated by two units in the positive y direction.

Write down

(i) the y-coordinates of the image points of *A* and *B* under this translation

(ii) the equations of the hyperbola and the asymptotes after the translation.

Only assessed at A-level

Teaching guidance

Students should be able to:

- determine the equation of a parabola, hyperbola or ellipse following a combination of two or more transformations of either a stretch parallel to one of the axes, a reflection in one of the axes, a reflection in $y = \pm x$, a translation, a rotation about the origin or an enlargement
- determine the transformations that have been applied by considering the equation.

Notes

Rotations will be limited to multiples of $\frac{\pi}{2}$ or $90°$

Examples

1 A hyperbola H has equation $x^2 - \frac{y^2}{2} = 1$

Describe a sequence of transformations which maps H onto the curve with equation $x^2 = 2(y^2 + x)$

2 An ellipse E has equation $\frac{x^2}{16} + \frac{y^2}{9} = 1$

(a) Sketch the ellipse E, showing the values of the intercepts on the coordinate axes.

(b) Given that the line with equation $y = x + k$ intersects the ellipse E at two distinct points, show that $-5 < k < 5$

(c) The ellipse E is translated by the vector $\begin{bmatrix} a \\ b \end{bmatrix}$ to form another ellipse whose equation is

$$9x^2 + 16y^2 + 18x - 64y = c$$

Find the values of the constants a, b and c

(d) **Hence** find an equation for each of the two tangents to the ellipse $9x^2 + 16y^2 + 18x - 64y = c$ that are parallel to the line $y = x$

E Further calculus

E1 Evaluate improper integrals where either the integrand is undefined at a value in the range of integration or the range of integration extends to infinity.

Only assessed at A-level

Teaching guidance

Students should be able to:

- identify why a given integral is improper
- understand and clearly demonstrate the limiting processes involved, using correct notation
- identify when and why an improper integral does not have a finite value
- identify the point(s) where the integrand is undefined
- evaluate improper integrals where the integrand is undefined at a value in between the limits of integration.

Examples

1 (a) Explain why the integral $\displaystyle\int_0^3 \frac{1}{(x-1)^{\frac{2}{3}}}\,\mathrm{d}x$ is improper

(b) Clearly showing the limiting process used, evaluate the integral.

2 For each of the following improper integrals, find the value of the integral **or** explain why it does not have a value:

(a) $\displaystyle\int_1^\infty x^{-\frac{3}{4}}\,\mathrm{d}x$

(b) $\displaystyle\int_1^\infty x^{-\frac{5}{4}}\,\mathrm{d}x$

(c) $\displaystyle\int_1^\infty \left(x^{-\frac{3}{4}} - x^{-\frac{5}{4}}\right)\mathrm{d}x$

3 (a) For each of the following improper integrals, find the value of the integral **or** explain briefly why it does not have a value:

(i) $\int_0^9 \frac{1}{\sqrt{x}}\,dx$

(ii) $\int_0^9 \frac{1}{x\sqrt{x}}\,dx$

(b) Explain briefly why the integrals in part (a) are improper integrals.

E2	Derive formulae for and calculate volumes of revolution.

Assessed at AS and A-level

Teaching guidance

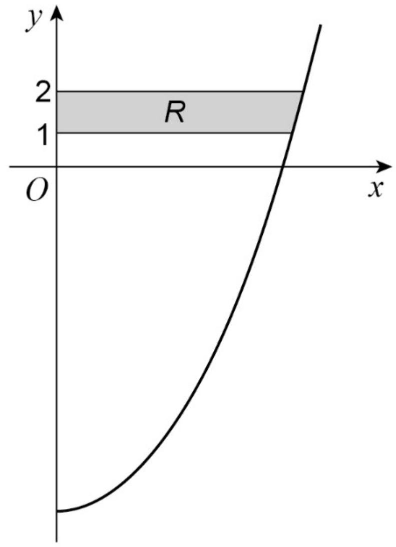
Students should be able to:

- understand when a volume of revolution is formed
- derive the formulae $V = \int_a^b \pi y^2 \mathrm{d}x$ and $V = \int_a^b \pi x^2 \mathrm{d}y$, for volumes formed by rotation around the x-axis or y-axis respectively, by splitting the volume into cylinders and demonstrating the limiting process involved
- use the formulae $V = \int_a^b \pi y^2 \mathrm{d}x$ and $V = \int_a^b \pi x^2 \mathrm{d}y$ to evaluate volumes of revolution and composite volumes.

Examples

1 Evaluate $\int_{-\infty}^{\infty} \frac{1}{x^2+1} \mathrm{d}x$, clearly showing the limiting process used.

2 The diagram shows the curve $y = 18x^2 - 9$ for $x \geq 0$



A solid is formed when the region R is rotated through 360° about the y-axis.

(a) By considering the volume of thin discs, show that the volume of the solid can be found using $V = \pi\int_1^2 y + 9 \,\mathrm{d}y$

(b) Find the exact value of the volume of the solid.

Only assessed at A-level

Teaching guidance

Students should be able to extend the use of the formulae $V = \int_a^b \pi y^2 \,\mathrm{d}x$ and $V = \int_a^b \pi x^2 \,\mathrm{d}y$ to evaluate integrals that require the use of integration techniques encountered at A-level.

Example

1 The region bounded by the curve $y = \left(5\sqrt{x}\right)\sec x$, the x-axis from 0 to 1 and the line $x = 1$ is rotated through 2π radians about the x-axis to form a solid.

Find the value of the volume of the solid generated, giving your answer to two significant figures.

E3

Understand and evaluate the mean value of a function.

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- understand and use the formula $y_M = \frac{1}{b-a}\int_a^b \mathrm{f}(x)\,\mathrm{d}x$ to calculate the mean value of a function for $x \in [a, b]$
- extend the use of the formula $y_M = \frac{1}{b-a}\int_a^b \mathrm{f}(x)\,\mathrm{d}x$ with functions that require integration techniques encountered at A-level.

Examples

1 Find the mean value of the function $\mathrm{f}(x) = x(1 - x)$ for $0 \le x \le 1$

2 The mean value of the function $\mathrm{f}(x) = x^3$ over the interval $x \in [a, b]$ is zero.

What can you deduce about a and b?

3 Find the mean value of the function $\mathrm{f}(x) = \sin x$ for $0 \le x \le \pi$

4 The mean value of the function $\mathrm{f}(x) = \sinh x$ over the interval $x \in [a, b]$ is zero.

What can you deduce about a and b?

E4

Integrate using partial fractions (extend to quadratic factors $ax^2 + c$ in the denominator).

Only assessed at A-level

Teaching guidance

Students should be able to:

- understand when partial fractions need to be used to complete an integral
- apply standard techniques to find integrals once partial fractions have been found
- complete the square to write an irreducible quadratic factor in the form $a(x + b)^2 + c$

Example

1 $$\frac{dy}{dx} = \frac{4x^2 + x + 7}{4x^3 - 8x^2 + 9x - 18}$$

Find an expression for y

E5

Differentiate inverse trigonometric functions.

Only assessed at A-level

Teaching guidance

Students should be able to:

- understand and use standard results:

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sin^{-1}x)=\frac{1}{\sqrt{1-x^2}}, \quad |x|<1$$
$$\frac{\mathrm{d}}{\mathrm{d}x}(\cos^{-1}x)=-\frac{1}{\sqrt{1-x^2}}, \quad |x|<1$$
$$\frac{\mathrm{d}}{\mathrm{d}x}(\tan^{-1}x)=\frac{1}{1+x^2}$$

- derive these results using implicit differentiation.

Note

The notation $\sin^{-1}$ or arcsin will be used.

Examples

1 $\quad \mathrm{f}(x)=\sin^{-1}\left(\dfrac{1}{\sqrt{x^2+1}}\right)$

Find $\mathrm{f}'(x)$ and state the values of x for which the derivative is valid.

2 (a) $y=\tan^{-1}x$

Show that $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{1}{1+x^2}$

(b) Differentiate $x\tan^{-1}x$ with respect to x

E6

Integrate functions of the form $\left(a^2 - x^2\right)^{\frac{1}{2}}$ and $\left(a^2 + x^2\right)^{-1}$ and be able to choose trigonometric substitutions to integrate associated functions.

Only assessed at A-level

Teaching guidance

Students should be able to:

- understand and use the standard results, given in the formulae book:

$$\int \frac{1}{a^2 + x^2}\,\mathrm{d}x = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,\mathrm{d}x = \sin^{-1}\left(\frac{x}{a}\right) + c\,,\ |x| < a$$

- derive these results using appropriate trigonometric substitutions
- write an integrand in a form to which these results can be applied
- complete the square to write an irreducible quadratic factor in the form $a(x + b)^2 + c$

Examples

1 Using an appropriate substitution, find $\displaystyle\int \frac{1}{4x^2 + 9}\,\mathrm{d}x$

2 (a) Show that $\displaystyle\int \frac{1}{a^2 + x^2}\,\mathrm{d}x = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$

(b) Find the exact value of $\displaystyle\int_{-2}^{\sqrt{3}-3} \frac{1}{x^2 + 6x + 12}\,\mathrm{d}x$

3 Using integration by substitution prove the result $\displaystyle\int \frac{1}{\sqrt{a^2 - x^2}}\,\mathrm{d}x = \sin^{-1}\left(\frac{x}{a}\right) + c\,,\ |x| < a$

E7 Arc length and area of surface of revolution for curves expressed in Cartesian or parametric coordinates.

Only assessed at A-level

Teaching guidance

Students should be able to:

- understand and use the formula $s = \int_a^b \sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$ for the arc length of a curve from the point where $x = a$ to the point where $x = b$ when the equation of the curve is given in Cartesian form
- understand and use the formula $s = \int_{t_1}^{t_2} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t$ for the arc length of a curve defined parametrically, where t_1 and t_2 are the values of the parameter at each end of the arc
- understand and use the formula $A = \int_a^b 2\pi y\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$ for the area of surface of revolution for a curve expressed in Cartesian form
- understand and use the formula $A = \int_{t_1}^{t_2} 2\pi y\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t$ for the area of surface of revolution for a curve expressed in parametric form.

Examples

1 The arc of the curve with equation $y = 4 - \ln(1 - x^2)$ from $x = 0$ to $x = \frac{3}{4}$ has length s.

(a) Show that $s = \int_0^{\frac{3}{4}} \left(\frac{1+x^2}{1-x^2}\right)\mathrm{d}x$

(b) Find the value of s, giving your answer in the form $p + \ln N$, where p is a rational number and N is an integer.

2 A curve is defined parametrically by $x = t^3 + 5, \quad y = 6t^2 - 1$

The arc length between the points where $t = 0$ and $t = 3$ on the curve is s.

(a) Show that $s = \int_0^3 3t\sqrt{t^2 + A}\,\mathrm{d}t$, stating the value of the constant A.

(b) Hence show that $s = 61$

3 The arc of the curve with equation $y = \frac{1}{2}\cos 4x$ between the points where $x = 0$ and $x = \frac{\pi}{8}$ is rotated through 2π radians about the x-axis.

(a) Show that the area s of the curved surface formed is given by

$s = \pi\int_0^{\frac{\pi}{8}} \cos 4x\sqrt{1 + 4\sin^2 4x}\,\mathrm{d}x$

(b) Use the substitution $u = \sin 4x$ to find the exact value of s.

4 A curve C is given parametrically by the equations $x = \frac{1}{2}\cosh 2t, \quad y = 2\sinh t$

(a) Express $\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2$ in terms of $\cosh t$

(b) The arc of C from $t = 0$ to $t = 1$ is rotated through 2π radians about the x-axis.

(i) Show that S, the area of the curved surface generated, is given by

$S = 8\pi\int_0^1 \sinh t\cosh^2 t\,\mathrm{d}t$

(ii) Find the exact value of S.

E8

Derivation and use of reduction formulae for integration.

Only assessed at A-level

Teaching guidance

Students should be able to:

- understand how reduction techniques can be used to repeatedly reduce the complexity of the integrand to enable a given integral to be calculated
- understand how integration by parts may be applied to produce reduction formulae
- understand how other techniques of integration may be used to find reduction formulae
- use reduction formulae to evaluate integrals.

Examples

1 Use a reduction formula to evaluate $\int_0^{\frac{\pi}{2}} \sin^{11} x \, \mathrm{d}x$

2 (a) If $I_n \equiv \int \cos^n x \, \mathrm{d}x$ use integration by parts to show that $I_n = \frac{1}{n} \sin x \cos^{n-1} x + \frac{n-1}{n} I_{n-2}$

(b) Hence find $\int \cos^5 x \, \mathrm{d}x$

3 (a) Show that $\tan^n x \equiv \sec^2 x \tan^{n-2} x - \tan^{n-2} x$

(b) Hence find the exact value of $\int_0^{\frac{\pi}{4}} \tan^6 x \, \mathrm{d}x$

E9

The limits $\lim\limits_{x\to\infty}\left(x^k e^{-x}\right)$ and $\lim\limits_{x\to 0}\left(x^k \ln x\right)$ where $k > 0$, applied to improper integrals.

Only assessed at A-level

Teaching guidance

Students should be able to:

- understand, prove, and use the result $\lim\limits_{x\to\infty}(x^k e^{-x}) = 0$
- understand and use the result $\lim\limits_{x\to 0}(x^k \ln x) = 0$, where $k > 0$, proof not required
- apply these results to evaluate improper integrals.

Examples

1 (a) Explain why $\int_2^\infty (x-2)e^{-2x}\,dx$ is an improper integral.

(b) Evaluate $\int_2^\infty (x-2)e^{-2x}\,dx$, showing the limiting process used.

2 (a) Explain why $\int_0^1 x^4 \ln x\,dx$ is an improper integral.

(b) Evaluate $\int_0^1 x^4 \ln x\,dx$, showing the limiting process used.

3 Prove $\lim\limits_{x\to\infty}(x^2 e^{-x}) = 0$

F Further vectors

F1 Understand and use the vector and Cartesian forms of an equation of a straight line in 3D.

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- understand and use the forms $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$, and $\dfrac{x - x_1}{l} = \dfrac{y - y_1}{m} = \dfrac{z - z_1}{n}$
- convert between the different forms
- given sufficient information, find the equation of a line.

Notes

Knowledge of 3D vectors from A-level Mathematics is assumed. This is additional content for AS Further Mathematics.

If any of l, m or n is equal to zero, the convention used will be to write, for example, $x = 2, \dfrac{y-3}{4} = \dfrac{z+1}{7}$ rather than $\dfrac{x-2}{0} = \dfrac{y-3}{4} = \dfrac{z+1}{7}$

Examples

1 Find the Cartesian equation of the line PQ where P and Q have coordinates (1, 3, 5) and (–1, 0, 4) respectively.

2 The quadrilateral $ABCD$ has vertices $A(2, 1, 3)$, $B(6, 5, 3)$, $C(6, 1, -1)$ and $D(2, -3, -1)$.

The line l_1 has vector equation $\mathbf{r} = \begin{bmatrix} 6 \\ 1 \\ -1 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

(a) (i) Find the vector $\overrightarrow{AB}$

(ii) Show that the line AB is parallel to l_1

(iii) Verify that D lies on l_1

(b) The line l_2 passes through D and $M(4, 1, 1)$.

(i) Find a vector equation of l_2

3 The line l passes through the point $(2, 7, 1)$ in the direction $2\mathbf{i} + 5\mathbf{k}$

(a) Find a vector equation of l

(b) Find a Cartesian equation of l

F2 Understand and use the vector and Cartesian forms of the equation of a plane.

Only assessed at A-level

Teaching guidance

Students should be able to:

- understand and use the forms $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$ and $ax + by + cz = d$
- convert between the different forms
- given sufficient information, find the equation of a plane
- understand the conditions under which two or more planes intersect and interpret geometrically
- understand and use links with matrices and determinants
- find the line of intersection of two planes
- find the point of intersection of three planes or the equation of their line of intersection.

Examples

1 Find a vector equation for the plane given by $2x + 3y - z = 6$

2 Find the Cartesian equation of the plane given by $\mathbf{r} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \mu \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

3 The plane Π contains the points (1, 0, 2), (2, 1, 3) and (3, 2, –1)

(a) Find a vector equation of Π

(b) Find the Cartesian equation of Π

(c) Determine which of the points $A(3, 2, -1)$, $B(2, 0, -2)$ and $C(1, 0, q)$ lies on Π

Fully justify your answer.

4 Show that the lines $\mathbf{r} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}$ and $\mathbf{r} = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ lie in the same plane and find the Cartesian equation of the plane.

5 Find the equation of the line of intersection of the planes $x + y - z = 7$ and $2x + 3y - 4z = 2$

6 Two planes have equations $2x - 2y + z = 24$ and $x + 3y + 4z = 8$

They meet in a line L.

Find the Cartesian equation of the line L.

7 A set of three planes is given by the system of equations

$$x + 3y - z = 10$$
$$2x + ky + z = -4$$
$$3x + 5y + (k - 2)z = k + 4$$

where k is a constant.

(a) Show that $\begin{vmatrix} 1 & 3 & -1 \\ 2 & k & 1 \\ 3 & 5 & k-2 \end{vmatrix} = k^2 - 5k + 6$

(b) In each of the following cases, determine the **number** of solutions of the given system of equations:

(i) $k = 1$

(ii) $k = 2$

(iii) $k = 3$

(c) Give a geometrical interpretation of the significance of each of the three results in part (b) in relation to the three planes.

F3

Calculate the scalar product and use it to calculate the angle between two lines, to express the equation of a plane, and to calculate the angle between two planes and the angle between a line and a plane.

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- understand and use the definition for two vectors **a** and **b**:

  $\mathbf{a}.\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$, where θ is the angle between **a** and **b**

- understand how the definition leads to $\begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} \cdot \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} = a_1a_2 + b_1b_2 + c_1c_2$

- find the acute or obtuse angle between two direction vectors.

Examples

1 The points A and B have coordinates (3, 2, 10) and (5, –2, 4) respectively.

The line l passes through A and has equation $\mathbf{r} = \begin{bmatrix} 3 \\ 2 \\ 10 \end{bmatrix} + \lambda \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$

Find the acute angle between l and the line segment AB.

2 The coordinates of the points A and B are (3, –2, 1) and (5, 3, 0) respectively.

The line l has equation $\mathbf{r} = \begin{bmatrix} 5 \\ 3 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$

(a) Find the distance between A and B.

(b) Find the acute angle between the lines AB and l. Give your answer to the nearest degree.

Only assessed at A-level

Teaching guidance

Students should be able to:

- understand and use the equation of a plane in the form **r . n** $= d$ and link to the Cartesian and vector equation of a plane
- use the fact that **n** is a normal to the plane **r . n** $= d$
- find the equation of the line of intersection of two planes
- use the fact that the angle between two planes is equal to the angle between their normals
- use the fact that the angle between a line and a plane is the complement of the angle between the line and the normal to the plane.

Examples

1 The plane Π_1 is perpendicular to the vector 9**i** – 8**j** + 72**k** and passes through the point *A*(2, 10, 1).

(a) Find, in the form **r . n** $= d$, a vector equation of Π_1

(b) Determine the exact value of the cosine of the acute angle between Π_1 and the plane Π_2 with equation **r**.(**i** + **j** + **k**) = 11

2 The line *L* and the plane Π are, respectively, given by the equations

$$\mathbf{r} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} \text{ and } \mathbf{r}.\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = 20$$

Determine the size of the acute angle between *L* and Π

3 The plane Π has equation $\mathbf{r} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} + \mu \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix}$

(a) Find an equation for Π in the form **r . n** $= d$

(b) Show that the line with equation $\mathbf{r} = \begin{bmatrix} 7 \\ 1 \\ 4 \end{bmatrix} + t \begin{bmatrix} 10 \\ 1 \\ 5 \end{bmatrix}$ does not intersect with Π, and explain the geometrical significance of this result.

4 A line and a plane have equations $\dfrac{x-3}{p} = \dfrac{y-q}{3} = \dfrac{z-1}{-1}$ and $\mathbf{r}.\begin{bmatrix}1\\1\\2\end{bmatrix} = 10$ respectively,

where p and q are constants.

(a) Show that the line is **not** perpendicular to the plane.

(b) In the case where the line lies in the plane, find the values of p and q.

(c) In the case where the angle, θ, between the line and the plane satisfies $\sin\theta = \dfrac{1}{\sqrt{6}}$ and the line intersects the plane at $z = 2$

(i) Find the value of p

(ii) Find the value of q

F4 Check whether vectors are perpendicular by using the scalar product.

Assessed at AS and A-level

Teaching guidance

Students should understand and use the result that two non-zero vectors **a** and **b** are perpendicular if and only if their scalar product is zero: $\mathbf{a} \cdot \mathbf{b} = 0$

Examples

1 The points A and B have coordinates (2, 4, 1) and (3, 2, –1) respectively. The point C is such that $\overrightarrow{OC} = 2\overrightarrow{OB}$, where O is the origin.

The point $P(\alpha, \beta, \gamma)$ is such that BP is perpendicular to AC.

Show that $4\alpha - 3\gamma = 15$

2 The lines l_1 and l_2 have equations $\mathbf{r} = \begin{bmatrix} 8 \\ 6 \\ -9 \end{bmatrix} + \lambda \begin{bmatrix} 3 \\ -3 \\ -1 \end{bmatrix}$ and $\mathbf{r} = \begin{bmatrix} -4 \\ 0 \\ 11 \end{bmatrix} + \mu \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ respectively.

(a) Show that l_1 and l_2 are perpendicular.

(b) Show that l_1 and l_2 intersect and find the coordinates of the point of intersection, P.

(c) The point $A(-4, 0, 11)$ lies on l_2. The point B on l_1 is such that $AP = BP$.

Find the length of AB.

3 Two vectors **a** and **b** are given by $\mathbf{a} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 4 \\ 2 \\ -2 \end{bmatrix}$

Show that **a** and **b** are perpendicular.

F5

Calculate and understand the properties of the vector product.

Understand and use the equation of a straight line in the form $(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = 0$

Use vector products to find the area of a triangle.

Only assessed at A-level

Teaching guidance

Students should be able to:

- understand and use the definition for two vectors **a** and **b**: $\mathbf{a}\times\mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\,\hat{\mathbf{n}}$, where θ is the angle between **a** and **b** and $\hat{\mathbf{n}}$ is the right-handed unit normal vector to **a** and **b**
- understand and use the fact that $\mathbf{a}\times\mathbf{b}$ is perpendicular to both **a** and **b**
- understand and use the algebraic properties of the vector product for example: $\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$ $\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = \mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}$
- calculate the vector product in component form
- understand and use the vector product form of the equation of a straight line: $(\mathbf{r}-\mathbf{a})\times\mathbf{b} = \mathbf{0}$, where **a** is the position vector of a point on the line and **b** is the direction vector of the line
- understand and use the fact that the area of a triangle with sides **a** and **b** can be found using area $= \frac{1}{2}|\mathbf{a}\times\mathbf{b}|$ and use this to solve problems involving area.

Examples

1 The vectors **a**, **b** and **c** are such that $\mathbf{c}\times\mathbf{a} = 2\mathbf{i}$ and $\mathbf{b}\times\mathbf{a} = 3\mathbf{j}$

Simplify $(\mathbf{a} + 2\mathbf{b} - 6\mathbf{c}) \times (\mathbf{a} - \mathbf{b} + 3\mathbf{c})$, giving your answer in the form $\lambda\mathbf{i} + \mu\mathbf{j}$

2 The points A, B, C and D have position vectors **a**, **b**, **c** and **d** respectively relative to the origin O,

where $\mathbf{a} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix}$ and $\mathbf{d} = \begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix}$

(a) Find $\overrightarrow{AB} \times \overrightarrow{AC}$

(b) The points A, B and C lie in the plane Π. Find a Cartesian equation for Π.

3 Given that $\mathbf{a} \times \mathbf{b} = 3\mathbf{i} + \mathbf{j} + \mathbf{k}$ and that $\mathbf{a} \times \mathbf{c} = -\mathbf{i} - 2\mathbf{j} + \mathbf{k}$, determine:

(a) $\mathbf{c} \times \mathbf{a}$

(b) $\mathbf{a} \times (\mathbf{b} + \mathbf{c})$

(c) $(\mathbf{a} \times \mathbf{b}) \,.\, (\mathbf{a} \times \mathbf{c})$

(d) $\mathbf{a} \,.\, (\mathbf{a} \times \mathbf{c})$.

4 Three vectors **u**, **v** and **w** are such that $\mathbf{u} \times \mathbf{v} = \mathbf{u} \times \mathbf{w}$, where $\mathbf{u} \neq \mathbf{0}$ and $\mathbf{v} \neq \mathbf{w}$

Show that $\mathbf{v} - \mathbf{w} = \lambda\mathbf{u}$, where λ is a scalar.

5 The three points A, B and C have coordinates (2, –1, 1), (4, 3, –2) and (3, 0, –3) respectively.

(a) Find $\overrightarrow{AB} \times \overrightarrow{AC}$

(b) Hence find the exact value of the area of triangle ABC.

6 The line l_1 has Cartesian equation $\frac{x-1}{4} = \frac{2-y}{2} = \frac{z+1}{3}$ and the line l_2 has vector equation

$\mathbf{r} = \begin{bmatrix} 4 \\ 3 \\ c \end{bmatrix} + \lambda \begin{bmatrix} 5 \\ -1 \\ 0 \end{bmatrix}$ where c is a constant.

The plane Π_1 contains the lines l_1 and l_2

(a) Show that an equation for the plane Π_1 is $x + 5y + 2z = d$, where d is an integer to be found.

(b) Find the value of c

(c) The plane Π_2 has equation $2x - y + 2z = 4$

(i) Find the acute angle between the planes Π_1 and Π_2, giving your answer to the nearest 0.1^0

(ii) Find an equation of the line of intersection of Π_1 and Π_2, giving your answer in the form $(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = 0$

F6

Find the intersection of two lines.

Find the intersection of a line and a plane.

Calculate the perpendicular distance between two lines, from a point to a line and from a point to a plane.

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- show that two lines intersect and find their point of intersection
- prove that two skew or parallel lines do not intersect
- understand and use the fact that the shortest distance between two lines is in a direction perpendicular to both lines
- understand and use the fact that the shortest distance between a line, l, and a point is along a line through the point, perpendicular to l

Notes

At A-level students could typically use the formula $d = \left| \frac{(\mathbf{a}-\mathbf{c})\cdot(\mathbf{b}\times\mathbf{d})}{|\mathbf{b}\times\mathbf{d}|} \right|$ for the shortest distance between two lines. At AS the method would not be expected to use the vector product, but if a student worked in this way, using the formula cited here, full marks would be given.

Examples

1 Find the distance of the point (1, 1, 2) from the line $\mathbf{r} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + \lambda \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$

2 The lines l_1 and l_2 have equations $\mathbf{r} = \begin{bmatrix} 8 \\ 6 \\ -9 \end{bmatrix} + \lambda \begin{bmatrix} 3 \\ -3 \\ -1 \end{bmatrix}$ and $\mathbf{r} = \begin{bmatrix} -4 \\ 0 \\ 11 \end{bmatrix} + \mu \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ respectively.

(a) Show that l_1 and l_2 are perpendicular.

(b) Show that l_1 and l_2 intersect and find the coordinates of the point of intersection, *P*.

(c) The point *A*(–4, 0, 11) lies on l_2. The point *B* on l_1 is such that $AP = BP$.

Find the length of *AB*.

3 The lines l_1 and l_2 have vector equations $\mathbf{r} = (2 + \lambda)\mathbf{i} + (-2 - \lambda)\mathbf{j} + (7 + \lambda)\mathbf{k}$ and

$\mathbf{r} = (4 + 4\mu)\mathbf{i} + (26 + 14\mu)\mathbf{j} + (-3 - 5\mu)\mathbf{k}$, respectively, where λ and μ are scalar parameters.

(a) The vector $\mathbf{n} = -\mathbf{i} + a\mathbf{j} + b\mathbf{k}$, where a and b are integers, is perpendicular to both l_1 and l_2
Find the value of a and the value of b

(b) The point P on l_1 and the point Q on l_2 are such that $\overrightarrow{PQ} = m\mathbf{n}$ for some scalar constant m

(i) Determine the value of m

(ii) Deduce the shortest distance between l_1 and l_2

4 The points A and B have coordinates (5, 1, –2) and (4, –1, 3) respectively.

The line l has equation $\mathbf{r} = \begin{bmatrix} -8 \\ 5 \\ -6 \end{bmatrix} + \mu \begin{bmatrix} 5 \\ 0 \\ -2 \end{bmatrix}$

(a) Find a vector equation of the line that passes through A and B.

(b) (i) Show that the line that passes through A and B intersects the line l, and find the coordinates of the point of intersection, P.

(ii) The point C lies on l such that triangle PBC has a right angle at B. Find the coordinates of C.

Only assessed at A-level

Teaching guidance

Students should be able to:

- solve simultaneous equations for a line and a plane, given in any form, to find the point of intersection
- use the normal to a plane to find the distance between a plane and a point
- solve problems involving points, lines and planes.

Examples

1 The point Q has position vector $\mathbf{q} = \begin{bmatrix} 7 \\ 4 \\ 6 \end{bmatrix}$

The plane Π has equation $\mathbf{r}.\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = 36$

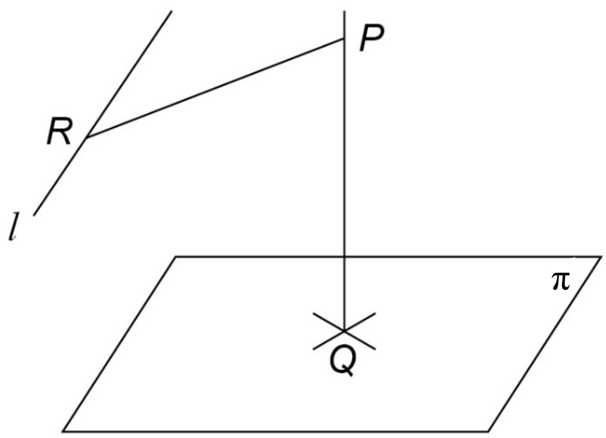
The line l has equation $\mathbf{r} = \begin{bmatrix} 20 \\ -8 \\ 1 \end{bmatrix} + \mu \begin{bmatrix} -7 \\ 5 \\ 3 \end{bmatrix}$

(a) Show that Q lies in Π.

(b) Show also that l is parallel to Π.

(c) The diagram shows that the point P, which lies on the normal to Π that passes through Q.

The point R is the point on l which is closest to P, and $PQ = PR$.



Determine the coordinates of P.

2 The line L has equation $\left(\mathbf{r}-\begin{bmatrix}2\\1\\0\end{bmatrix}\right)\times\begin{bmatrix}3\\-1\\2\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

The plane Π contains the line L and the point $A(4, 1, -2)$.

(a) Show that A does not lie on the line L.

(b) Find an equation of the plane Π, giving your answer in the form $\mathbf{r}\,.\,\mathbf{n}=c$

(c) The point D has coordinates $(8, -2, 6)$. Find the coordinates of the image of D after reflection in the plane Π.

3 The four vertices of a parallelogram $ABCD$ have coordinates $A(1, 0, 2)$, $B(3, -1, 5)$, $C(7, 2, 4)$ and $D(5, 3, 1)$

(a) (i) Find $\overrightarrow{AB}\times\overrightarrow{AD}$

(ii) Show that the area of the parallelogram is $p\sqrt{10}$, where p is an integer to be found.

(b) The diagonals AC and BD of the parallelogram meet at the point M. The line L passes through M and is perpendicular to the plane $ABCD$.

Find an equation for the line L, giving your answer in the form $(\mathbf{r}-\mathbf{u})\times\mathbf{v}=0$

(c) The plane Π is parallel to the plane $ABCD$ and passes through the point $Q(6, 5, 17)$.

(i) Find the coordinates of the point of intersection of the line L with the plane Π.

(ii) Find the distance between the planes Π and $ABCD$.

G Polar coordinates

G1 Understand and use polar coordinates and be able to convert between polar and Cartesian coordinates.

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- understand the terms 'pole', 'initial line' and the convention that angles will be in radians taken as positive in the anticlockwise direction from the initial line about the pole. (Knowledge of radians is assumed from A-level Maths.)
- describe a point using polar coordinates in the form (r, θ) and understand that the polar coordinates of a given point cannot be described uniquely,

  for example the point $\left(-2, \frac{\pi}{6}\right)$ can also be described as $\left(2, -\frac{5\pi}{6}\right)$ or $\left(2, \frac{7\pi}{6}\right)$
- understand and use the relationship between polar and Cartesian coordinates:

  if the initial line is taken to be the x-axis and the pole is at the origin, then

$$x = r\cos\theta \qquad\qquad r^2 = x^2 + y^2$$

or

$$y = r\sin\theta \qquad\qquad \tan\theta = \frac{y}{x}$$

- understand what is meant by a half-line in the cases when r is restricted to r ≥ 0
- solve problems using polar coordinates.

Examples

1 Find the polar equation of the circle of radius a, whose centre has polar coordinates $\left(a, \frac{\pi}{2}\right)$

2 The points A and B have polar coordinates $\left(\sqrt{3}, \frac{\pi}{3}\right)$ and $\left(1, -\frac{\pi}{6}\right)$ respectively.

Show that AB is perpendicular to the initial line, and find the length of AB.

3 Find the polar coordinates of the points with Cartesian coordinates

(a) (2, 2)

(b) $\left(-1, \sqrt{3}\right)$

(c) (–3, –4).

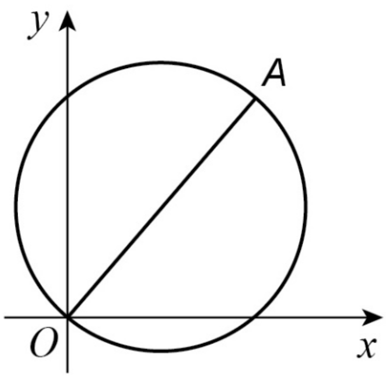
4 The curve C has the polar equation

$$r = \frac{2}{3 + 2\cos\theta}, \qquad 0 \le \theta \le 2\pi$$

(a) Verify that the point L with polar coordinates $(2, \pi)$ lies on C.

(b) Find the Cartesian equation of C, giving your answer in the form $9y^2 = f(x)$

5 The diagram shows a sketch of a circle which passes through the origin O.



The equation of the circle is $(x - 3)^2 + (y - 4)^2 = 25$ and OA is a diameter.

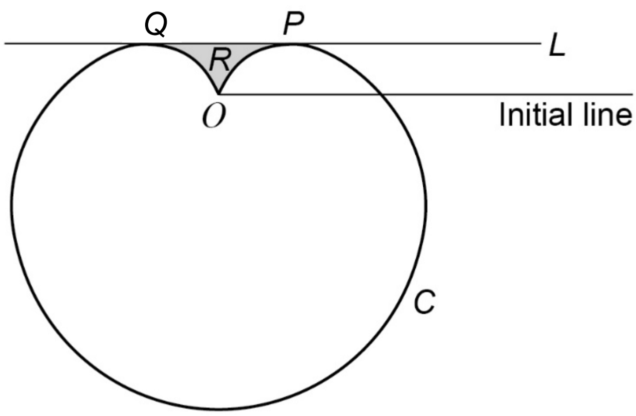
(a) Find the Cartesian coordinates of the point A.

(b) Using O as the pole and the positive x-axis as the initial line, the polar coordinates of A are (k, α).

(i) Find the value of k and the value of $\tan\alpha$

(ii) Find the polar equation of the circle $(x - 3)^2 + (y - 4)^2 = 25$, giving your answer in the form $r = p\cos\theta + q\sin\theta$

6 The diagram shows a sketch of a curve C and a line L, which is parallel to the initial line and touches the curve at the points P and Q.



The polar equation of the curve C is $r = 4(1 - \sin\theta)$, $0 \leq \theta < 2\pi$

and the polar equation of the line L is $r \sin\theta = 1$

Show that the polar coordinates of P are $\left(2, \frac{\pi}{6}\right)$ and find the polar coordinates of Q.

7 The points A and B have polar coordinates $\left(3, \frac{\pi}{6}\right)$ and $\left(4, -\frac{\pi}{3}\right)$ respectively.

(a) Show that $AB = 5$

(b) Find the Cartesian coordinates of A and B.

(c) Use the Cartesian coordinates of A and B to verify that $AB = 5$

8 The points A and B have polar coordinates $\left(2, \frac{\pi}{6}\right)$ and $\left(3, -\frac{\pi}{2}\right)$ respectively.

(a) Find the angle between OA and OB, where O is the pole.

(b) Use the cosine rule to find the distance between A and B.

G2

Sketch curves with r given as a function of θ, including use of trigonometric functions.

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- use a given function to find coordinates (r, θ), which can be plotted
- use symmetries of functions to aid sketching. For example, a function of cos θ will be symmetrical about the line $\theta = 0$ and a function of sin θ will be symmetrical about the line $\theta = \frac{\pi}{2}$
- use the property that if $r \to 0$ as $\theta \to \alpha$, then the line $\theta = \alpha$ will be a tangent to the curve at the pole.

Examples

1 Sketch the curve with equation $r = 2a \cos 2\theta$

2 (a) Sketch the curve with the polar equation

$$r = 1 - \sin\theta, \qquad -\pi < \theta \le \pi$$

(b) State the polar equation of the tangent to the curve at the pole.

3 A line l and a curve C have polar equations

$$r\sin\theta = 2, \qquad r = \frac{2}{1+\sin\theta} \qquad 0 < \theta \le \pi$$

(a) Sketch l and C on the same diagram.

(b) The point P, with polar coordinates (a, ϕ), lies on C and O is the pole. The foot of the perpendicular from P onto l is N.

Show that $OP = PN$.

4 (a) Sketch the curve with the polar equation

$$r = 2\sin 3\theta, \qquad -\pi < \theta \le \pi$$

(b) Give the polar equations of the tangents to the curve at the pole.

G3

Find the area enclosed by a polar curve.

Only assessed at A-level

Teaching guidance

Students should be able to:

- understand and use the formula $A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, \mathrm{d}\theta$ for a polar curve defined as $r = \mathrm{f}(\theta)$ for $\alpha \le \theta \le \beta$
- use symmetry to simplify the evaluation of areas
- find composite areas.

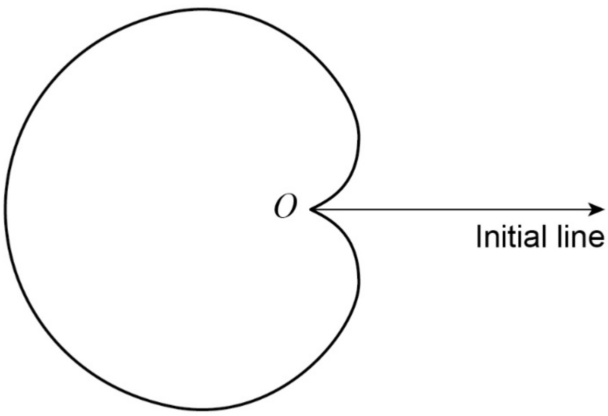
Examples

1 (a) Write down the polar equation of a circle of radius a with centre at the pole O.

(b) Use the formula $A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \mathrm{d}\theta$ to show that the area of the circle is πa^2

2 The diagram shows the curve C with polar equation

$$r = 6(1 - \cos\theta) \;, \qquad 0 \le \theta < 2\pi$$



(a) Find the area of the region bounded by the curve C.

(b) The circle with Cartesian equation $x^2 + y^2 = 9$ intersects the curve C at the points A and B.

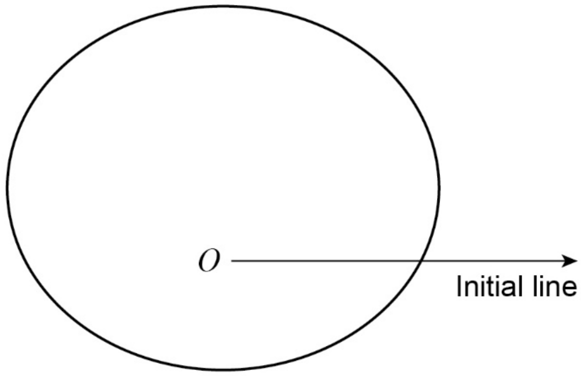
(i) Find the polar coordinates of A and B.

(ii) Find, in surd form, the length of AB.

3 (a) A circle C_1 has Cartesian equation $x^2 + (y - 6)^2 = 36$

Show that the polar equation of C_1 is $r = 12 \sin \theta$

(b) A curve C_2 with the polar equation $r = 2 \sin \theta + 5$, $0 \leq \theta \leq 2\pi$ is shown in the diagram.

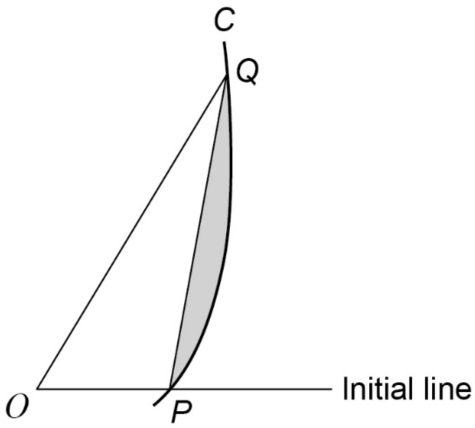


Calculate the area bounded by C_2

(c) The circle C_1 intersects the curve C_2 at the points P and Q.

Find, in surd form, the area of the quadrilateral $OPMQ$, where M is the centre of the circle and O is the pole.

4 The diagram shows a sketch of part of the curve C whose polar equation is $r = 1 + \tan \theta$. The point O is the pole.



The points P and Q on the curve are given by $\theta = 0$ and $\theta = \dfrac{\pi}{3}$ respectively.

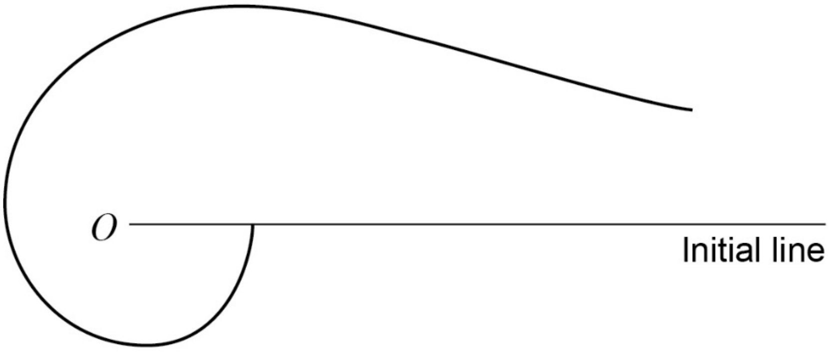
(a) Show that the area of the region bounded by the curve C and the lines OP and OQ is

$$\frac{1}{2}\sqrt{3} + \ln 2$$

(b) Hence find the area of the shaded region bounded by the line PQ and the arc PQ of C.

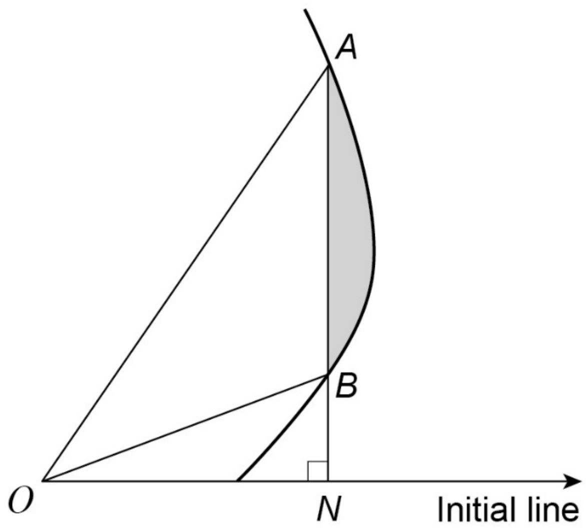
5 The diagram shows a sketch of the curve whose polar equation is

$$r = \theta^{-\frac{1}{2}}, \qquad 0 < \theta \leq 2\pi$$



Show that the area enclosed between the curve and the lines $\theta = \alpha$ and $\theta = 2\alpha$, where $0 < \alpha \leq \pi$ is independent of α

6 The diagram shows the sketch of part of a curve, the pole O and the initial line.



The polar equation of the curve is $r = 1 + \tan\theta$

The point A is the point on the curve at which $\theta = \dfrac{\pi}{3}$

The perpendicular, AN, from A to the initial line intersects the curve at the point B.

(a) Find the exact length of OA.

(b) (i) Given that, at the point B, $\theta = \alpha$, show that $(\cos\alpha + \sin\alpha)^2 = 1 + \dfrac{\sqrt{3}}{2}$

(ii) Hence, or otherwise, find α in terms of π

(c) Show that the area of triangle OAB is $\dfrac{3+2\sqrt{3}}{6}$

(d) Find, in an exact simplified form, the area of the shaded region bounded by the curve and the line segment AB.

7 (a) Show that $(\cos\theta + \sin\theta)^2 = 1 + \sin 2\theta$

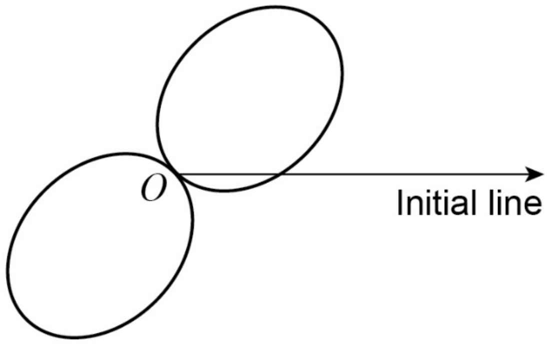
(b) A curve has Cartesian equation $(x^2 + y^2)^3 = (x + y)^4$

Given that $r \geq 0$, show that the polar equation of the curve is $r = 1 + \sin 2\theta$

(c) The curve with polar equation

$$r = 1 + \sin 2\theta, \qquad -\pi \leq \theta \leq \pi$$

is shown in the diagram below.



(i) Find the two values of θ for which $r = 0$

(ii) Find the area of one of the loops.

AQA

H Hyperbolic functions

H1

Understand the definitions of hyperbolic functions sinh x, cosh x and tanh x, including their domains and ranges, **and be able to sketch their graphs.**

Understand the definitions of hyperbolic functions sech x, cosech x and coth x, including their domains and ranges.

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- recall and use the definitions

  $$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2}$$

- recall the definition $\tanh x = \dfrac{\sinh x}{\cosh x}$ and derive the exponential form $\tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$
- use the exponential forms to solve equations or prove identities
- sketch graphs of $y = \sinh x$, $y = \cosh x$ and $y = \tanh x$ (including asymptotes)
- apply single transformations to graphs of hyperbolic functions.

Examples

1 Prove that $\sinh x + \cosh x > 0$ for all values of x

2 By sketching suitable graphs, show that the equation $\cosh x - \tanh x = 1$ has two solutions.

3 Using the definition of $\tanh x$ in terms of $\sinh x$ and $\cosh x$ prove that $\tanh x = \dfrac{(e^x - 1)(e^x + 1)}{e^{2x} + 1}$

4 (a) Show that the equation 14sinh x – 10cosh x = 5, can be expressed as $2e^{2x} - 5e^{x} - 12 = 0$

(b) Hence solve the equation 14sinh x – 10cosh x = 5

Give your answer as a natural logarithm.

5 (a) Using the definition $\sinh\theta = \frac{1}{2}(e^{\theta} - e^{-\theta})$, prove the identity $4\sinh^3\theta + 3\sinh\theta = \sinh 3\theta$

(b) Given that $x = \sinh\theta$ and $16x^3 + 12x - 3 = 0$, find the value of θ in terms of a natural logarithm.

(c) Hence find the real root of the equation $16x^3 + 12x - 3 = 0$, giving your answer in the form $2^p - 2^q$, where p and q are rational numbers.

6 Use the definitions $\sinh\theta = \frac{1}{2}(e^{\theta} - e^{-\theta})$ and $\cosh\theta = \frac{1}{2}(e^{\theta} + e^{-\theta})$ to show that:

(a) $2\sinh\theta\cosh\theta = \sinh 2\theta$

(b) $\cosh^2\theta + \sinh^2\theta = \cosh 2\theta$

Only assessed at A-level

Teaching guidance

Students should be able to:

- recall and use the definitions
  $\text{sech}\, x = \dfrac{1}{\cosh x}$ $\quad \text{cosech}\, x = \dfrac{1}{\sinh x}$ $\quad \coth x = \dfrac{1}{\tanh x}$
- sketch graphs of all hyperbolic functions
- apply transformations to hyperbolic functions
- derive exponential forms of these hyperbolic functions
- recall and use the domains and ranges for all hyperbolic functions.

Examples

1 The function f is defined by $\mathrm{f}(x) = \tanh x,\ x \in \mathbb{R}$

State the range of f

2 Prove that $\text{sech}\, x + \text{cosech}\, x = 2\mathrm{e}^{x}\text{cosech}\, 2x$

3 The function f is defined by $\mathrm{f}(x) = \text{sech}\, x$

(a) State the maximum possible domain of f

(b) State the range of f

(c) Sketch the graph of $y = \mathrm{f}(x)$

(d) Sketch the graph of $y = 2 - 2\text{sech}\, x$

H2 Differentiate and integrate hyperbolic functions.

Teaching guidance

Only assessed at A-level

Students should be able to:

- recall and use the derivatives $\frac{d}{dx}(\sinh x) = \cosh x$ and $\frac{d}{dx}(\cosh x) = \sinh x$; using the logarithmic definitions derive these results.
- use standard results:

$$\frac{d}{dx}(\tanh x) = \text{sech}^2 x \qquad \frac{d}{dx}(\text{cosech } x) = -\text{cosech } x \coth x$$

$$\frac{d}{dx}(\text{sech } x) = -\text{sech } x \tanh x \qquad \frac{d}{dx}(\coth x) = -\text{cosech}^2 x$$

  and derive results using standard techniques such as the product rule, quotient rule and chain rule
- recall and use the results $\int \sinh x \, dx = \cosh x + c$ and $\int \cosh x \, dx = \sinh x + c$
- use any of the techniques of integration covered in Maths or Further Maths to integrate functions involving hyperbolic functions.

Examples

1 Differentiate the following expressions:

(a) $\cosh 3x$ (b) $\cosh^2 3x$ (c) $x^2 \cosh x$

(d) $\frac{\cosh 2x}{x}$ (e) $x \tanh x$ (f) $\text{sech } x$

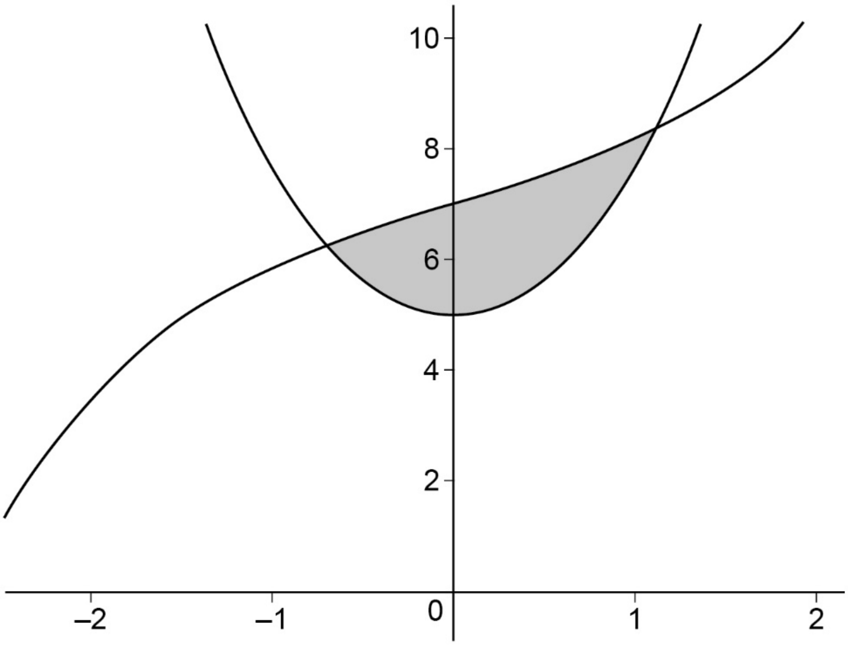
(g) $\text{cosech } x$

2 Evaluate the following integrals:

(a) $\int \cosh 3x \, dx$ (b) $\int \cosh^2 x \, dx$

(c) $\int x \sinh 2x \, dx$ (d) $\int \tanh^2 x \, dx$

3 The diagram shows the graphs of $y = 5\cosh x$ and $y = 7 + \sinh x$



Find the exact value of the shaded area.

4 A curve has Cartesian equation $y = \frac{1}{2}\ln(\tanh x)$

(a) Show that $\frac{dy}{dx} = \frac{1}{\sinh 2x}$

(b) The points *A* and *B* on the curve have x-coordinates $\ln 2$ and $\ln 4$ respectively.

Find the arc length *AB*, giving your answer in the form $p \ln q$, where $p, q \in \mathbb{Q}$

5 (a) Prove that the curve $y = 12\cosh x - 8\sinh x - x$ has exactly one stationary point.

(b) Given that the coordinates of this stationary point are (a, b), show that $a + b = 9$

6 (a) Given that $y = \ln \tanh \frac{x}{2}$, where $x > 0$, show that $\frac{dy}{dx} = \operatorname{cosech} x$

(b) A curve has equation $y = \ln \tanh \frac{x}{2}$, where $x > 0$. The length of the arc of the curve between the points where $x = 1$ and $x = 2$ is denoted by s

(i) Show that $s = \int_1^2 \coth x \, dx$

(ii) Hence show that $s = \ln(2\cosh 1)$.

H3 Understand and be able to use the definitions of the inverse hyperbolic functions and their domains and ranges.

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- understand and use the inverse hyperbolic functions
- understand both notations arsinh x and $\sinh^{-1} x$
- solve equations involving hyperbolic functions
- sketch the graphs of inverse hyperbolic functions, including intersections with the axes, asymptotes and transformations.

Notes

The inverse function arsinh is sometimes erroneously written as arcsinh and the same applies to all inverse hyperbolic functions. Students will not be penalised for using arcsinh.

Examples

1 Using a suitable sketch, show that the equation $\cosh x = 1 + \operatorname{arsinh} x$ has exactly two solutions.

2 Solve the equation $6\cosh^2 x - 7\cosh x - 5 = 0$

Give your answers to 3 significant figures.

3 Sketch the graph of $y = \operatorname{artanh}(x - 1)$

Only assessed at A-level

Teaching guidance

Students should be able to:

- recall and use the domains and ranges of inverse hyperbolic functions
- understand the domain restriction required for $\cosh^{-1} x$

Examples

1 Sketch the graph with the equation y = arcosh $(2x)$ and state the range of values of x for which arcosh $(2x)$ is defined.

2 Sketch the graph with the equation $y = \text{artanh}\left(\frac{x}{2}\right)$

3 The function f is given by $\mathrm{f}(x) = \text{artanh}\, x$

(a) Sketch the graph with the equation $y = \mathrm{f}(x)$

(b) State the maximum domain of f and the corresponding range.

H4 Derive and use the logarithmic forms of the inverse hyperbolic functions.

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- use the logarithmic forms:

  $\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1})$

  $\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$

  $\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$

- derive these results.

Examples

1 Given that $u = \tanh x$, use the definitions of $\sinh x$ and $\cosh x$ in terms of e^x and e^{-x} to show that $x = \frac{1}{2}\ln\left(\frac{1+u}{1-u}\right)$

2 Express arsinh $(x^2 - 1)$ in terms of natural logarithms.

3 Solve the equation $\operatorname{artanh}\left(\frac{x^2-1}{x^2+1}\right) = \ln 2$

Only assessed at A-level

Teaching guidance

Students should be able to derive logarithmic formulae for sech, cosech and coth.

Example

1 Express arsech x in terms of natural logarithms.

AQA

H5

Integrate functions of the form $\left(x^2+a^2\right)^{-\frac{1}{2}}$ and $\left(x^2-a^2\right)^{-\frac{1}{2}}$ and be able to choose substitutions to integrate associated functions.

Only assessed at A-level

Teaching guidance

Students should be able to:

- recognise when a hyperbolic substitution would be useful in evaluating an integral
- differentiate inverse hyperbolic functions.

Examples

1 A curve has equation $y=4\sqrt{x}$

(a) Show that the length of arc s of the curve between the points where x = 0 and x = 1 is given by $s=\int_0^1\sqrt{\frac{x+4}{x}}\,\mathrm{d}x$

(b) (i) Use the substitution $x=4\sinh^2\theta$ to show that $\int\sqrt{\frac{x+4}{x}}\,\mathrm{d}x=\int 8\cosh^2\theta\,\mathrm{d}\theta$

(ii) Hence show that $s=4\sinh^{-1}0.5+\sqrt{5}$

2 By making a suitable substitution find the integral $\int\sqrt{x^2+4x+5}\,\mathrm{d}x$

3 (a) Show that $\frac{\mathrm{d}}{\mathrm{d}x}\left(\cosh^{-1}\frac{1}{x}\right) = \frac{-1}{x\sqrt{1-x^2}}$

(b) A curve has equation $y = \sqrt{1-x^2} - \cosh^{-1}\frac{1}{x}$ $(0 < x < 1)$

Show that:

(i) $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sqrt{1-x^2}}{x}$

(ii) the length of the arc of the curve from the point where $x = \frac{1}{4}$ to the point where $x = \frac{3}{4}$ ln 3

H6

Understand and use $\tanh x \equiv \dfrac{\sinh x}{\cosh x}$

Understand and use $\cosh^2 x - \sinh^2 x \equiv 1$,
$\text{sech}^2 x \equiv 1 - \tanh^2 x$, $\text{cosech}^2 x \equiv \coth^2 x - 1$,
$\cosh 2x \equiv \cosh^2 x + \sinh^2 x$ and $\sinh 2x \equiv 2\sinh x \cosh x$

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- recall and use the identities:

  $\tanh x \equiv \dfrac{\sinh x}{\cosh x}$

  $\cosh^2 x - \sinh^2 x \equiv 1$
- derive the identity $\cosh^2 x - \sinh^2 x \equiv 1$
- use identities to solve equations or derive other results.

Example

1 (a) Given that $4\cosh^2 x = 7\sinh x + 1$, find the two possible values of $\sinh x$

(b) Hence obtain the two possible values of x, giving your answers in the form $\ln p$

Only assessed at A-level

Teaching guidance

Students should be able to:

- recall and use the identities:

  $\text{sech}^2\theta \equiv 1 - \tanh^2\theta$

  $\text{cosech}^2\theta \equiv \coth^2\theta - 1$
- use the identities:

  $\cosh 2x \equiv \cosh^2 x + \sinh^2 x$

  $\sinh 2x \equiv 2\sinh x \cosh x$
- derive these identities
- use identities to solve equations or derive other results
- use identities to evaluate integrals.

Examples

1 (a) Use the definitions of $\sinh x$ and $\cosh x$ in terms of e^x and e^{-x} to show that $\text{sech}^2 x + \tanh^2 x = 1$

(b) Solve the equation $6\text{sech}^2 x = 4 + \tanh x$, giving your answers in terms of natural logarithms.

2 (a) Use the definitions $\sinh\theta = \frac{1}{2}(e^\theta - e^{-\theta})$ and $\cosh\theta = \frac{1}{2}(e^\theta + e^{-\theta})$ to show that $1 + 2\sinh^2\theta = \cosh 2\theta$

(b) Solve the equation $3\cosh 2\theta = 2\sinh\theta + 11$ giving each of your answers in the form $\ln p$

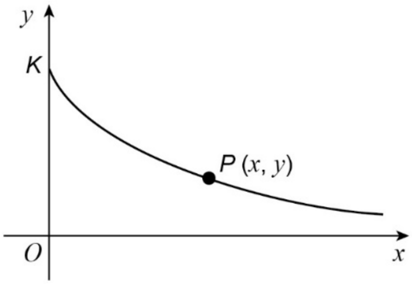
3 (a) Given that y = sech t, show that:

(i) $\frac{dy}{dt} = -\text{sech}\, t\ \tanh t$

(ii) $\left(\frac{dy}{dt}\right)^2 = \text{sech}^2\, t - \text{sech}^4\, t$

(b) The diagram shows a sketch of part of the curve given parametrically by:

$x = t - \tanh t \qquad y = \text{sech}\, t$



The curve meets the y-axis at the point K, and $P(x, y)$ is a general point on the curve.

The arc length KP is denoted by s

Show that:

(i) $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \tanh^2 t$

(ii) $s = \ln \cosh t$

(iii) $y = e^{-s}$

4 (a) Use the definition $\cosh x = \frac{1}{2}(e^x + e^{-x})$ to show that $\cosh 2x = \cosh^2 x - 1$

(b) (i) The arc of the curve $y = \cosh x$ between $x = 0$ and $x = \ln a$ is rotated through 2π radians about the x-axis.

Show that S, the surface area generated, is given by $S = 2\pi \int_0^{\ln a} \cosh^2 x \, dx$

(ii) Hence show that $S = \pi\left(\ln a + \frac{a^4 - 1}{4a^2}\right)$

5 (a) Given that $u = \cosh^2 x$, show that $\frac{du}{dx} = \sinh 2x$

(b) Hence show that $\int_0^1 \frac{\sinh 2x}{1 + \cosh^4 x} dx = \tan^{-1}(\cosh^2 1) - \frac{\pi}{4}$

H7 Construct proofs involving hyperbolic functions and identities.

Only assessed at A-level

Teaching guidance

Students should be able to:

- use the results in H6 in proofs.

Example

1 Prove

$$\frac{1}{2}\sinh x \sinh 2x \equiv \cosh^3 x - \cosh x$$

I Differential equations

I1

Find and use an integrating factor to solve differential equations of form $\frac{\mathrm{d}y}{\mathrm{d}x} + \mathrm{P}(x)y = \mathrm{Q}(x)$ and recognise when it is appropriate to do so.

Only assessed at A-level

Teaching guidance

Students should be able to:

- recognise when a given differential equation may be rearranged into the form $\frac{\mathrm{d}y}{\mathrm{d}x} + \mathrm{P}(x)y = \mathrm{Q}(x)$ to allow an integrating factor to be used
- find an integrating factor by inspection or by using the formula $I = \mathrm{e}^{\int \mathrm{P}(x)\,\mathrm{d}x}$
- use the result $\int \left(I\frac{\mathrm{d}y}{\mathrm{d}x} + I\mathrm{P}(x)y \right) \mathrm{d}x = Iy$
- solve more difficult differential equations by transforming them into this form by means of a suitable given substitution.

Examples

1 Solve the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} + (\cot x)y = 2\cos x$, given that y = 2 when $x = \frac{\pi}{2}$

2 (a) Show that $\frac{1}{x}$ is an integrating factor for the first-order differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{1}{x}y = x\ln x$$

(b) Solve this differential equation, given that $y = 1$ when $x = 2$

(c) Calculate the value of y when $x = 1.2$, giving your answer to three decimal places.

3 Solve the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{2}{x}y = 3(x^3 + 1)^{\frac{1}{2}}$, given that $y = 1$ when $x = 2$

4 (a) A differential equation is given by $x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - \frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2$

Show that the substitution $u = \frac{\mathrm{d}y}{\mathrm{d}x}$ transforms this differential equation into $\frac{\mathrm{d}u}{\mathrm{d}x} - \frac{1}{x}u = 3x$

(b) By using an integrating factor, find the general solution of $\frac{\mathrm{d}u}{\mathrm{d}x} - \frac{1}{x}u = 3x$

Give your answer in the form $u = \mathrm{f}(x)$

(c) Hence find the general solution of the differential equation $x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - \frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2$

Give your answer in the form $y = \mathrm{g}\left(x\right)$

I2 Find both general and particular solutions of differential equations.

Only assessed at A-level

Teaching guidance

Students should be able to:

- understand that the general solution comprises the complementary function + a particular integral
- find general solutions to differential equations including the appropriate number of constants of integration
- find particular solutions to differential equations
- use boundary or initial conditions, that are given explicitly or need to be deduced from the context of the question, to find constants of integration.

Examples

1 Solve the differential equation $\frac{dy}{dx} + 2y = 2x^2 + 3$, given that $y(0) = 5$

2 (a) Show that $y = x^3 - x$ is a particular integral of the differential equation $\frac{dy}{dx} + \frac{2xy}{x^2 - 1} = 5x^2 - 1$

(b) By differentiating $(x^2 - 1)y = c$ implicitly, where y is a function of x and c is a constant, show that $y = \frac{c}{x^2 - 1}$ is a solution of the differential equation $\frac{dy}{dx} + \frac{2xy}{x^2 - 1} = 0$

(c) Hence find the general solution of $\frac{dy}{dx} + \frac{2xy}{x^2 - 1} = 5x^2 - 1$

3 (a) Find the values of the constants a, b, c and d for which $a + bx + c \sin x + d \cos x$ is a particular integral of the differential equation $\frac{dy}{dx} - 3y = 10 \sin x - 3x$

(b) Hence find the general solution of this differential equation.

I3 Use differential equations in modelling in kinematics and in other contexts.

Only assessed at A-level

Teaching guidance

Students should be able to:

- interpret questions set in a variety of contexts including, but not limited to, kinematics, population growth, science and finance
- make appropriate modelling assumptions and clearly define their variables
- understand the language of proportionality and rates of change, rates of increase and rates of decrease
- apply their knowledge of kinematics from A-level Maths.

Examples

1 A frozen ready meal with a temperature of –15°C is placed in an oven at room temperature.

The oven is switched on so that its temperature at time t minutes is given by $20 + 10t$.

The temperature of the ready meal increases at a rate which is proportional to the difference between the temperature of the oven and the temperature of the ready meal. Initially the temperature of the ready meal is increasing at a rate of 0.35°C per minute.

Find the temperature of the ready meal when it has been in the oven for 20 minutes.

2 A small town has a population of y thousand people.

The town's population has a natural rate of growth proportional to its size so that its rate of increase is given by $0.02y$.

In 2010, the population of the town was 250 000, and over the next few years people moved away from the town at a rate of $0.4 + e^{0.2t}$ thousand people per year, where t is the time in years from 2010.

(a) Write down a first order differential equation to model the total rate of change of population.

(b) Use your model to predict the size of the population at the end of 2015.

14

Solve differential equations of form $y'' + ay' + by = 0$ where a and b are constants, by using the auxiliary equation.

Only assessed at A-level

Teaching guidance

Students should be able to:

- understand that the more general form $ay'' + by' + cy = 0$ is equivalent to an equation of the form $y'' + ay' + by = 0$
- know that a 2nd order differential equation of this form is called a homogeneous equation
- use $ay'' + by' + cy = 0$ to form the auxiliary equation $ak^2 + bk + c = 0$
- use a substitution of the form $y = e^{kx}$ to derive the auxiliary equation
- understand how the solutions of the auxiliary equation can be used to form the general solution of the corresponding differential equation.

Examples

1 Find the solution of the differential equation $4\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 5y = 0$ satisfying conditions that $y = 1$ and $\frac{dy}{dx} = 0$ when $x = 0$

2 (a) Find the general solution of the homogeneous differential equation $4\frac{d^2y}{dx^2} - y = 0$

(b) Hence find the particular solution which is such that $y = 1$ and $\frac{dy}{dx} = 0$ when $x = 0$

3 Find the function $y(x)$ satisfying the differential equation $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = 0$ and the conditions that $y = 0$ and $\frac{dy}{dx} = 1$ when $x = 0$

4 Show that if y_1 and y_2 are solutions of the differential equation $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$, then so also is the linear combination $y = Ay_1 + By_2$, where A and B are arbitrary constants.

I5

Solve differential equations of form $y'' + ay' + by = \mathrm{f}(x)$ where a and b are constants, by solving the homogeneous case and adding a particular integral to the complementary function (in cases where $\mathrm{f}(x)$ is a polynomial, exponential or trigonometric function).

Only assessed at A-level

Teaching guidance

Students should be able to:

- understand that the more general form $ay'' + by' + cy = \mathrm{f}(x)$ is equivalent to an equation of the form $y'' + ay' + by = \mathrm{f}(x)$
- solve the homogeneous equation $ay'' + by' + cy = 0$ to find the complementary function (CF)
- find the particular integral (PI) for a given $\mathrm{f}(x)$,where $\mathrm{f}(x)$ could be formed from the sum of polynomial, exponential or trigonometric terms
- form the general solution using $y = \text{CF} + \text{PI}$

Examples

1 Find the solution of the differential equation $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4y = 10\mathrm{e}^{4x} + 8\sin 2x + 4\cos 2x$

given that y = 2.5 when x = 0 and $y = \frac{\pi}{4}$ when $x = \frac{\pi}{4}$

2 (a) Find the roots of the equation $m^2 + 2m + 2 = 0$ in the form $a + \mathrm{i}b$

(b) (i) Find the general solution of the differential equation $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 4x$

(ii) Hence express y in terms of x, given that y = 1 and $\frac{\mathrm{d}y}{\mathrm{d}x} = 2$ when $x = 0$

3 It is given that y satisfies the differential equation $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 4y = 8x - 10 - 10\cos 2x$

(a) Show that $y = 2x + \sin 2x$ is a particular integral of the given differential equation.

(b) Find the general solution of the differential equation.

(c) Hence express y in terms of x, given that $y = 2$ and $\frac{dy}{dx} = 0$ when $x = 0$

4 The function $y(x)$ satisfies the differential equation $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = e^{3x}$

(a) Find the complementary function.

(b) Show that there is a particular integral of the form $y = ax^2e^{3x}$, and find the appropriate value of a

(c) Hence write down the general solution for $y(x)$

16

Understand and use the relationship between the cases when the discriminant of the auxiliary equation is positive, zero and negative and the form of solution of the differential equation.

Only assessed at A-level

Teaching guidance

Students should be able to, given $ay''+by'+cy=0$, examine the nature of the roots of the auxiliary equation to determine the form of the general solution:

If $b^2 - 4ac > 0$ then $y = Ae^{\alpha x} + Be^{\beta x}$ where α and β are the distinct real roots

If $b^2 - 4ac = 0$ then $y = e^{\alpha x}(A + Bx)$ where α is the repeated root

If $b^2 - 4ac < 0$ then $y = e^{px}(A\cos qx + B\sin qx)$ where $p \pm qi$ are the complex roots.

Examples

1 Find the general solution of the differential equation $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$

2 Find the general solution of the differential equation $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0$

3 Find the general solution of the differential equation $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 0$

Express the answer in a form involving trigonometric functions.

4 (a) Show that when $b^2 = 4ac$, the roots of the auxiliary equation for the differential equation $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$ are both equal to $-\frac{b}{2a}$

(b) Writing $-\frac{b}{2a}$ as k_1, verify that, in this case, $e^{k_1 x}$ and $xe^{k_1 x}$ are solutions of the differential equation.

I7	Solve the equation for simple harmonic motion $\ddot{x} = -\omega^2 x$ and relate the solution to the motion.

Only assessed at A-level

Teaching guidance

Students should be able to:

- understand the conditions for simple harmonic motion (SHM)
- show that a particle moves with simple harmonic motion by demonstrating that the equation $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\omega^2 x$ holds
- recall and use the standard results:

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\omega^2 x \Rightarrow \quad \begin{aligned} &v = \omega\sqrt{a^2 - x^2} \\ &x = a\cos \omega t \text{ or} \\ &x = a\sin \omega t \\ &T = \frac{2\pi}{\omega} \end{aligned}$$

  where a is the maximum distance from the equilibrium position and T is the periodic time

- solve the differential equation to derive these results
- understand and use the result $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} = v\frac{\mathrm{d}v}{\mathrm{d}x}$

Notes

The motion will always begin at the equilibrium position or at the maximum displacement from this position.

Examples

1 A particle moves in a straight line with simple harmonic motion such that its displacement at time t seconds relative to a fixed origin on this line is x metres.

The motion of the particle satisfies the differential equation $\dfrac{d^2x}{dt^2}+16x=0$

(a) Verify that $x = A\cos 4t + B\sin 4t$, where A and B are constants, is a solution to this differential equation.

(b) When $t = 0$, the particle is momentarily at rest. Show that $B = 0$

(c) Given that $x = h$ ($h > 0$) when $t = \dfrac{\pi}{2}$, find A in terms of h

(d) Find the maximum speed of the particle in terms of h

(e) The mass of the particle is m kg.

Find the magnitude of the maximum force acting on the particle during the motion.

Give your answer in terms of h and m

2 A particle moves along a straight line between the points A and B with simple harmonic motion. The point O is the mid-point of AB.

At time t seconds, the particle is x metres from O and moving with speed v ms^{-1}.

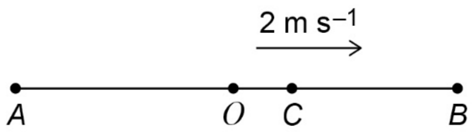
The motion of the particle satisfies the equation $\frac{d^2x}{dt^2} + \omega^2 x = 0$

(a) Given that A and B are a distance of 0.8 metres apart and by writing $\frac{d^2x}{dt^2}$ as $v\frac{dv}{dx}$ show that v can be written as $v = \omega\sqrt{0.16 - x^2}$

(b) By writing v as $\frac{dx}{dt}$ and separating the variables, find an expression for x in terms of ω and t

(c) Given that the particle completes one oscillation in 7.5 seconds, show that $\omega = \frac{4\pi}{15}$

(d) When the particle passes through the point C, as shown in the diagram, $x = 1$ and $v = 2$



(i) Show that the amplitude of the motion is 2.59 metres, correct to three significant figures.

(ii) When the particle first passes through C, it is heading away from O towards B.

Find the time that it takes to move from C to B and back to C, giving your answer to two significant figures.

(iii) Find the maximum speed of the particle during the oscillations.

I8

Model damped oscillations using 2nd order differential equations and interpret their solutions. Understand light, critical, and heavy damping and be able to determine when each will occur.

Only assessed at A-level

Teaching guidance

Students should be able to:

- set up and solve second order differential equations which represent damped harmonic motion
- understand the meaning of heavy damping (also known as over-damping), critical damping and light damping (also known as under-damping)
- recognise equations of the form $\frac{d^2x}{dt^2}+k\frac{dx}{dt}+\omega^2x=0$ as a model for damped motion and understand that the type of damping depends on the value of $k^2-4\omega^2$
- distinguish the following cases:

$k^2-4\omega^2>0$	$x=Ae^{\alpha t}+Be^{\beta t}$ where α and β are roots of the auxiliary equation	Heavy damping
$k^2-4\omega^2=0$	$x=(A+Bt)e^{-\frac{k}{2}t}$	Critical damping
$k^2-4\omega^2<0$	$x=e^{-\frac{k}{2}t}(A\sin\frac{\alpha}{2}t+B\cos\frac{\alpha}{2}t)$ where $\alpha^2=4\omega^2-k^2$	Light damping

Examples

1 A uniform metal bar, of mass m, is held at rest in a horizontal position. The ends of the bar are attached to identical light elastic strings.

The strings are also attached to fixed points that are directly above the ends of the bar. A damping device is also connected to the bar.

The bar is released from rest with the strings vertical and at their natural length.

As the bar falls, it remains horizontal and the damping device exerts an upward force of magnitude cmv on the centre of the bar, where c is a constant, in appropriate units, and v is the speed of the bar.

The motion of the bar is critically damped.

At time t after the bar has been released, the displacement of the bar below its initial position is x and the tension, T in each spring is given by $T = -2mx$

(a) Show that $c = 4$

(b) Find an expression for x in terms of g and t

(c) Find the value of x as t tends to infinity.

(d) Find the maximum speed of the bar.

2 A railway truck, of mass m, is travelling in a straight line along a horizontal track.

At time $t = 0$, the truck strikes one end of a buffer which is fixed at its other end.

The buffer may be modelled as a light spring.

At time t, the compression of the buffer is x and the magnitude of the force, T , in the spring is given by $T = mn^2 x$, where n is a positive constant.

The motion of the truck is affected by a resistance force of magnitude mkv , where v is the speed of the truck and k is a positive constant.

(a) Show that, while the buffer is bring compressed, x satisfies the equation $\ddot{x} + k\dot{x} + n^2 x = 0$

(b) A time $t = 0$, the truck is travelling with speed U. Given that $k = \frac{5n}{2}$, find x in terms of n, U and t

(c) By means of a sketch, or otherwise, explain whether the type of damping is light, critical or heavy.

I9

Analyse and interpret models of situations with one independent variable and two dependent variables as a pair of coupled 1st order simultaneous equations and be able to solve them, for example predator-prey models.

Only assessed at A-level

Teaching guidance

Students should be able to:

- interpret a given context to form a pair of first order simultaneous differential equations of the form $\frac{dx}{dt} = ax + by + f(t)$ and $\frac{dy}{dt} = cx + dy + g(t)$ where t is the independent variable and x and y are the dependent variables
- eliminate x or y to form and solve a second order differential equation of no greater demand than those outlined in section I5
- solve a pair of first order simultaneous differential equations to find the dependent variables as functions of the independent variable.

Examples

1 The variables x, y and t are linked by the differential equations

$$\frac{dy}{dt} = y - 2x + \sin t \quad \text{and} \quad \frac{dx}{dt} = \cos t - 3x + 4y$$

(a) Solve these equations to find the general solution for x in terms of t

(b) Find the general solution for y in terms of t

2 An isolated island is populated by rabbits and foxes. At time the number of rabbits is x and the number of foxes is y

It is assumed that:

- The number of foxes increases at a rate proportional to the number of rabbits. When there are 200 rabbits the number of foxes is increasing at a rate of 25 foxes per unit period of time.
- If there were no foxes present, the number of rabbits would increase by 115% in a unit period of time.
- When both foxes and rabbits are present the foxes kill rabbits at a rate that is equal to 105% of the current number of foxes.
- At time t = 0, the number of foxes is 10 and the number of rabbits is 60.

By setting up two differential equations, construct a mathematical model for the number of rabbits at time t

3 Two large tanks, labelled A and B, contain salt water.

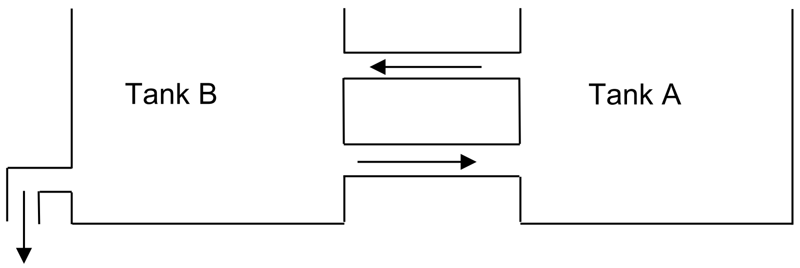
Each tank has a capacity of 1200 litres. Initially, tank A holds 600 litres of water with 15 grams of salt dissolved in it and tank B contains 1200 litres of water with 96 grams of salt dissolved.

Salt water is pumped into tank A at a rate of 8 litres per hour, with a concentration of 1 gram of salt per litre.

Fresh water is pumped into tank B at a rate of 3 litres per hour.

Through two connecting pipes water flows from tank A to tank B at a rate of 12 litres per hour and from B to A at a rate of 4 litres per hour.

Water is drained from tank B at a rate of 11 litres per hour so that the water in each tank remains constant.



Find expressions for the amount of salt in each tank in terms of time.

I10

Use of Hooke's law with $T = kx$ to formulate a differential equation for simple harmonic motion, where k is a constant.

Only assessed at A-level

Teaching guidance

Students should be able to:

- understand and use the model for tension given by Hooke's law in the form $T = kx$ where the tension is always directed toward the equilibrium point.
- understand what is meant by the terms natural length and extension of an elastic string or spring
- understand that this model will produce differential equations with terms of the form:
  $m\dfrac{d^2x}{dt^2} = -kx$, where m is the mass of the particle being acted upon and k is a positive constant.

Examples

1 A railway truck, of mass m, is travelling in a straight line along a horizontal track.

At time $t = 0$, the truck strikes one end of a buffer which is fixed at its other end. The buffer may be modelled as a light spring.

At time t, the compression of the buffer is x and the magnitude of the force, T in the spring is given by $T = mn^2x$, where n is a positive constant.

In a simple model of the motion, the only force affecting the truck during this motion is the thrust from the buffer.

(a) Show that, while the truck is in contact with the buffer, the truck performs simple harmonic motion.

(b) Find, in terms of n, the period of this motion.

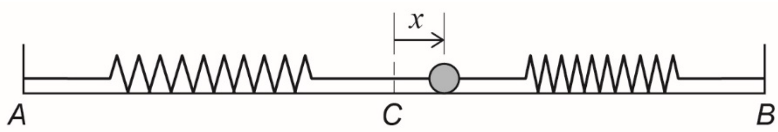
2 A particle, of mass 9 kg, is attached to two identical springs.

The other ends of the springs are attached to fixed points, *A* and *B*, which are 1.2 metres apart on a smooth horizontal surface.

The springs have natural length 0.4 m and the magnitude of the tension in each spring is given by $\frac{225e}{2}$, where e is the extension of the spring.

The particle is released from rest at a distance of 0.5 metres from *B* and moves on the line *AB*. The midpoint of *AB* is *C*.

At time t seconds after release, the displacement of the particle from *C* is x metres, where the direction from *A* to *B* is taken to be positive.



(a) Show that the resultant force on the particle, at time t, is $-225x$ newtons.

(b) Hence show that the particle moves with simple harmonic motion.

(c) State the period of this motion.

(d) Find the speed of the particle when it is 0.05 metres from *C*.

(e) Write down an expression for x in terms of t

I11 Use models for damped motion where damping force is proportional to the velocity.

Only assessed at A-level

Teaching guidance

Students should be able to:

- understand the damping force will act in the opposite direction to the velocity
- understand models of this type will lead to differential equations with terms of the form: $m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -k\frac{\mathrm{d}x}{\mathrm{d}t}$, where m is the mass of the particle being acted upon and k is a positive constant.

Examples

1 A particle *P*, of mass m kg, moves in a straight horizontal line.

At time t seconds, the displacement of *P* from a fixed point *O* on the line is x metres, and *P* is moving with velocity $\dot{x}$ ms^{-1}.

Throughout the motion, two horizontal forces act on *P*: a force of magnitude $4mn^2|x|$ newtons directed towards *O*, and a resistance force of magnitude $2mk|\dot{x}|$ newtons, where n and k are positive constants.

2 (a) Show that $\ddot{x} + 2k\dot{x} + 4n^2x = 0$

(b) In one case, $k = n$. When $t = 0$, $x = a$ and $\dot{x} = 0$

(i) Show that $x = \mathrm{e}^{-nt}\left(a\cos\sqrt{3}nt + \left(\frac{\sqrt{3}a}{3}\right)\sin\sqrt{3}nt\right)$

(ii) Show that *P* passes through *O* when $\tan\sqrt{3}nt = -\sqrt{3}$

(c) In a different case, $k = 2n$

(i) Find a general solution for x at time t seconds.

(ii) Hence state the type of damping which occurs.

3 In this question, use $g = 10$ ms^{-2}

A bungee jumper has mass 75 kg. He uses an elastic rope and falls vertically. When the rope becomes taut for the first time, he is travelling at 12.5 m s^{-1}

Assume that, once the rope is taut, the bungee jumper experiences an air resistance force that has magnitude $15v$ newtons, where v m s^{-1} is his speed.

At time t seconds after the rope has become taut, the extension of the rope is x metres and the tension in the rope has a magnitude of $\frac{75x}{2}$ N

(a) Show that $10\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2\frac{\mathrm{d}x}{\mathrm{d}t} + 5x = 100$

(b) Find x in terms of t

(c) Find the value of t when the bungee jumper first comes instantaneously to rest.

J Numerical methods

J1 Mid-ordinate rule and Simpson's rule for integration.

Only assessed at A-level

Teaching guidance

Students should be able to:

- understand and use the formula for the mid-ordinate rule:

$$\int_a^b y\,\mathrm{d}x \approx h\left(y_{\frac{1}{2}} + y_{\frac{3}{2}} + \ldots + y_{n-\frac{3}{2}} + y_{n-\frac{1}{2}}\right), \text{ where } h = \frac{b-a}{n}$$

- understand the circumstances under which the mid-ordinate rule gives an over- or underestimate
- understand and use the formula for Simpson's rule:

$$\int_a^b y\,\mathrm{d}x \approx \frac{1}{3}h\{(y_0 + y_n) + 4(y_1 + y_3 + \ldots + y_{n-1}) + 2(y_2 + y_4 + \ldots + y_{n-2})\}$$

where $h = \dfrac{b-a}{n}$ and n is even.

- understand how an estimate given by either method may be improved upon by increasing the number of strips.

Examples

1 Use Simpson's rule with 5 ordinates to find an approximation to $\displaystyle\int_1^3 \frac{1}{\sqrt{1+x^3}}\,\mathrm{d}x$

giving your answer to three significant figures.

2 (a) Use Simpson's rule with 7 ordinates to find an approximation to $\displaystyle\int_{0.5}^{1.7} (5 - x^x)\,\mathrm{d}x$ giving your answer to three significant figures.

(b) Hence find an approximation to $\displaystyle\int_{0.5}^{1.7} x^x\,\mathrm{d}x$

3 (a) Use the mid-ordinate rule with four strips of equal width to find an estimate for $\int_1^5 \frac{1}{1+\ln x}\,\mathrm{d}x$, giving your answer to three significant figures.

(b) Explain how you could find an improved estimate.

4 (a) Use the mid-ordinate rule with five strips of equal width to find an estimate for $\int_2^3 \cosh x\,\mathrm{d}x$ giving your answer to three significant figures.

(b) Using a sketch of the graph of $y = \cosh x$ explain clearly whether your answer to part (a) is an overestimate or an underestimate.

J2 Euler's step by step method for solving first order differential equations.

Only assessed at A-level

Teaching guidance

Students should be able to:

- apply Euler's formula, $y_{r+1} = y_r + h\mathrm{f}(x_r, y_r)$ to find approximate solutions to first order differential equations of the form $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(x)$ or $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(x, y)$
- understand the derivation of Euler's formula using a geometrical method
- understand how the accuracy of such a method may be improved by reducing the step size.

Examples

1 The function $y(x)$ satisfies the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(x, y)$ where $\mathrm{f}(x, y) = \frac{y - x}{y^2 + x}$ and $y(1) = 2$

Use the Euler formula $y_{r+1} = y_r + h\mathrm{f}(x_r, y_r)$ with $h = 0.1$, to obtain an approximation to $y(1.1)$

2 (a) The function $y(x)$ satisfies the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(x, y)$ where $\mathrm{f}(x, y) = x + 3 + \sin y$ and $y(1) = 1$

(b) Use the Euler formula $y_{r+1} = y_r + h\mathrm{f}(x_r, y_r)$ with $h = 0.1$, to obtain an approximation to $y(1.1)$, giving your answer to four decimal places.

(c) Explain how you could use this formula to find an improved approximation to $y(1.1)$

J3	Improved Euler method for solving first order differential equations.

Only assessed at A-level

Teaching guidance

Students should be able to:

- understand ways in which Euler's formula can be improved
- use given improved methods to find approximate solutions to first order differential equations of the form $\frac{dy}{dx} = f(x)$ or $\frac{dy}{dx} = f(x, y)$

Notes

There are many ways in which Euler's formula can be improved. A particular example is given in the formulae booklet. When an improved method is used in an exam question it will be clearly specified and could be different from the method given in the formulae book; we expect students to be able to use and interpret a recursive algorithm of this type.

Examples

1 It is given that $y(x)$ satisfies the differential equation $\frac{dy}{dx} = f(x, y)$

where $f(x, y) = (2x+1)\ln(x+y)$ and $y(0) = 2$

Use the improved Euler formula $y_{r+1} = y_r + \frac{1}{2}(k_1 + k_2)$ where $k_1 = hf(x_r, y_r)$ and $k_2 = hf(x_r + h, y_r + k_1)$ and $h = 0.1$, to obtain an approximation to $y(0.1)$, giving your answer to three decimal places.

2 The function $y(x)$ satisfies the differential equation $\frac{dy}{dx} = f(x, y)$ where $f(x, y) = \sqrt{x^2 + y + 1}$ and $y(3) = 2$

(a) Use the Euler formula $y_{r+1} = y_r + hf(x_r, y_r)$ with $h = 0.1$, to obtain an approximation to $y(3.1)$, giving your answer to four decimal places.

(b) Use the formula $y_{r+1} = y_{r-1} + 2hf(x_r, y_r)$ with your answer to part (a), to obtain $y(3.2)$, giving your answer to three decimal places.

A1 Appendix 1

Mathematical notation for AS and A-level qualifications in Maths and Further Maths

The tables below set out the notation that must be used by AS and A-level Mathematics and Further Mathematics specifications. Students will be expected to understand this notation without need for further explanation.

AS students will be expected to understand notation that relates to AS content, and will not be expected to understand notation that relates only to A-level content.

1	Set notation	
1.1	$\in$	is an element of
1.2	$\notin$	is not an element of
1.3	$\subseteq$	is a subset of
1.4	$\subset$	is a proper subset of
1.5	$\{x_1, x_2, \ldots\}$	the set with elements $x_1, x_2, \ldots$
1.6	$\{x : \ldots\}$	the set of all x such that …
1.7	$n(A)$	the number of elements in set A
1.8	$\varnothing$	the empty set
1.9	ε	the universal set
1.10	A'	the complement of the set A
1.11	$\mathbb{N}$	the set of natural numbers, $\{1, 2, 3, \ldots\}$
1.12	$\mathbb{Z}$	the set of integers, $\{0, \pm 1, \pm 2, \pm 3, \ldots\}$
1.13	$\mathbb{Z}^+$	the set of positive integers, $\{1, 2, 3, \ldots\}$
1.14	$\mathbb{Z}_0^+$	the set of non-negative integers, $\{0, 1, 2, 3, \ldots\}$

1.15	$\mathbb{R}$	the set of real numbers
1.16	$\mathbb{Q}$	the set of rational numbers, $\left\{\frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{Z}^{+}\right\}$
1.17	$\cup$	union
1.18	$\cap$	intersection
1.19	(x, y)	the ordered pair x, y
1.20	$[a, b]$	the closed interval $\{x \in \mathbb{R} : a \le x \le b\}$
1.21	$[a, b)$	the interval $\{x \in \mathbb{R} : a \le x < b\}$
1.22	$(a, b]$	the interval $\{x \in \mathbb{R} : a < x \le b\}$
1.23	(a, b)	the open interval $\{x \in \mathbb{R} : a < x < b\}$

1	Set notation (Further Mathematics only)	
1.24	$\mathbb{C}$	the set of complex numbers

2	Miscellaneous symbols	
2.1	$=$	is equal to
2.2	$\neq$	is not equal to
2.3	$\equiv$	is identical to or is congruent to
2.4	$\approx$	is approximately equal to
2.5	∞	infinity
2.6	$\propto$	is proportional to
2.7	$\therefore$	therefore
2.8	$\because$	because
2.9	$<$	is less than
2.10	$\le$	is less than or equal to, is not greater than

2.11	>	is greater than
2.12	$\geq$	is greater than or equal to, is not less than
2.13	$p \Rightarrow q$	p implies q (if p then q)
2.14	$p \Leftarrow q$	p is implied by q (if q then p)
2.15	$p \Leftrightarrow q$	p implies and is implied by q (p is equivalent to q)
2.16	a	first term of an arithmetic or geometric sequence
2.17	l	last term of an arithmetic sequence
2.18	d	common difference of an arithmetic sequence
2.19	r	common ratio of a geometric sequence
2.20	S_n	sum to n terms of a sequence
2.21	S_∞	sum to infinity of a sequence

3	Operations	
3.1	$a + b$	a plus b
3.2	$a - b$	a minus b
3.3	$a \times b$, ab, $a.b$	a multiplied by b
3.4	$a \div b$, $\frac{a}{b}$	a divided by b
3.5	$\sum_{i=1}^{n} a_i$	$a_1 + a_2 + \ldots a_n$
3.6	$\prod_{i=1}^{n} a_i$	$a_1 \times a_2 \times \ldots a_n$
3.7	$\sqrt{a}$	the non-negative square root of a
3.8	$\lvert a \rvert$	the modulus of a
3.9	$n!$	n factorial: $n! = n \times (n-1) \times \ldots \times 2 \times 1$, $n \in \mathbb{N}$; $0! = 1$

3.10	$\begin{pmatrix} n \\ r \end{pmatrix}$, ${}^{n}\mathrm{C}_{r}$, ${}_{n}\mathrm{C}_{r}$	the binomial coefficient $\dfrac{n!}{r!(n-r)!}$ for $n, r \in \square_0^+, r \le n$ or $\dfrac{n!(n-1)\ldots(n-r+1)}{r!}$ for $n \in \square$, $r \in \square_0^+$

4	Functions	
4.1	$\mathrm{f}(x)$	the value of the function f at x
4.2	$\mathrm{f} : x \mapsto y$	the function f maps the element x to the element y
4.3	f^{-1}	the inverse function of the function f
4.4	gf	the composite function of f and g which is defined by $\mathrm{gf}(x) = \mathrm{g}(\mathrm{f}(x))$
4.5	$\lim\limits_{x \to a} \mathrm{f}(x)$	the limit of $\mathrm{f}(x)$ as x tends to a
4.6	Δx, δx	an increment of x
4.7	$\dfrac{\mathrm{d}y}{\mathrm{d}x}$	the derivative of y with respect to x
4.8	$\dfrac{\mathrm{d}^n y}{\mathrm{d}x^n}$	the nth derivative of y with respect to x
4.9	$\mathrm{f}'(x), \mathrm{f}''(x), \ldots, \mathrm{f}^{(n)}(x)$	the first, second, …, nth derivatives of $\mathrm{f}(x)$ with respect to x
4.10	$\dot{x}, \ddot{x}, \ldots$	the first, second, … derivatives of x with respect to t
4.11	$\int y \, \mathrm{d}x$	the indefinite integral of y with respect to x
4.12	$\int_a^b y \, \mathrm{d}x$	the definite integral of y with respect to x between the limits $x = a$ and $x = b$

5	Exponentials and logarithmic functions	
5.1	e	base of natural logarithms
5.2	e^x, $\exp x$	exponential function of x
5.3	$\log_a x$	logarithm to the base a of x
5.4	$\ln x$, $\log_e x$	natural logarithm of x

6	Trigonometric functions	
6.1	$\left.\begin{array}{l}\sin, \cos, \tan \\ \operatorname{cosec}, \sec, \cot\end{array}\right\}$	the trigonometric functions
6.2	$\left.\begin{array}{l}\sin^{-1}, \cos^{-1}, \tan^{-1} \\ \arcsin, \arccos, \arctan\end{array}\right\}$	the inverse trigonometric functions
6.3	$\circ$	degrees
6.4	rad	radians

6	Trigonometric and hyperbolic functions (Further Mathematics only)	
6.5	$\left.\begin{array}{l}\operatorname{cosec}^{-1}, \sec^{-1}, \cot^{-1} \\ \operatorname{arccosec}, \operatorname{arcsec}, \operatorname{arccot}\end{array}\right\}$	the inverse trigonometric functions
6.6	$\left.\begin{array}{l}\sinh, \cosh, \tanh \\ \operatorname{cosech}, \operatorname{sech}, \coth\end{array}\right\}$	the hyperbolic functions
6.7	$\left.\begin{array}{l}\sinh^{-1}, \cosh^{-1}, \tanh^{-1} \\ \operatorname{cosech}^{-1}, \operatorname{sech}^{-1}, \coth^{-1}\end{array}\right\}$ $\left.\begin{array}{l}\operatorname{arsinh}, \operatorname{arcosh}, \operatorname{artanh} \\ \operatorname{arcosech}, \operatorname{arsech}, \operatorname{arcoth}\end{array}\right\}$	the inverse hyperbolic functions

7	Complex numbers (Further Mathematics only)	
7.1	i	square root of –1
7.2	$x + \mathrm{i}y$	complex number with real part x and imaginary part y
7.3	$r(\cos\theta + \mathrm{i}\sin\theta)$	modulus-argument form of a complex number with modulus r and argument θ
7.4	z	a complex number, $z = x + \mathrm{i}y = r(\cos\theta + \mathrm{i}\sin\theta)$
7.5	$\mathrm{Re}(z)$	the real part of z, $\mathrm{Re}(z) = x$
7.6	$\mathrm{Im}(z)$	the imaginary part of z, $\mathrm{Im}(z) = y$
7.7	$\lvert z \rvert$	the modulus of z, $\lvert z \rvert = \sqrt{x^2 + y^2}$
7.8	$\arg(z)$	the argument of z, $\arg(z) = \theta$, $-\pi < \theta \leq \pi$
7.9	z^*	the complex conjugate of z, $x - \mathrm{i}y$

8	Matrices (Further Mathematics only)	
8.1	$\mathbf{M}$	a matrix $\mathbf{M}$
8.2	$\mathbf{0}$	zero matrix
8.3	$\mathbf{I}$	identity matrix
8.4	$\mathbf{M}^{-1}$	the inverse of the matrix $\mathbf{M}$
8.5	$\mathbf{M}^{\mathrm{T}}$	the transpose of the matrix $\mathbf{M}$
8.6	Δ, det $\mathbf{M}$ or $\lvert\mathbf{M}\rvert$	the determinant of the square matrix $\mathbf{M}$
8.7	$\mathbf{Mr}$	image of column vector $\mathbf{r}$ under the transformation associated with the matrix $\mathbf{M}$

9	Vectors	
9.1	$\mathbf{a}, \underline{a}, \overline{a}, \underset{\sim}{a}, \tilde{a}$	the vector $\mathbf{a}, \underline{a}, \overline{a}, \underset{\sim}{a}, \tilde{a}$; these alternatives apply throughout section 9
9.2	$\overrightarrow{AB}$	the vector represented in magnitude and direction by the directed line segment AB

9.3	$\hat{\mathbf{a}}$	a unit vector in the direction of **a**
9.4	**i**, **j**, **k**	unit vectors in the directions of the Cartesian coordinate axes
9.5	$\lvert\mathbf{a}\rvert$, a	the magnitude of **a**
9.6	$\lvert\overrightarrow{AB}\rvert$, AB	the magnitude of $\overrightarrow{AB}$
9.7	$\begin{pmatrix} a \\ b \end{pmatrix}$, $a\mathbf{i}+b\mathbf{j}$	column vector and corresponding unit vector notation
9.8	$\mathbf{r}$	position vector
9.9	$\mathbf{s}$	displacement vector
9.10	$\mathbf{v}$	velocity vector
9.11	$\mathbf{a}$	acceleration vector

9	Vectors (Further Mathematics only)	
9.12	**a.b**	the scalar product of **a** and **b**

10	Differential equations (Further Mathematics only)	
10.1	ω	angular speed

11	Probability and statistics	
11.1	A, B, C, etc	events
11.2	$A \cup B$	union of the events A and B
11.3	$A \cap B$	intersection of the events A and B
11.4	$\mathrm{P}(A)$	probability of the event A
11.5	A'	complement of the event A
11.6	$\mathrm{P}(A \mid B)$	probability of the event A conditional on the event B
11.7	X, Y, R, etc	random variables
11.8	x, y, r, etc	values of the random variables X, Y, R, etc

11.9	$x_1, x_2, \ldots$	values of observations
11.10	$f_1, f_2, \ldots$	frequencies with which the observations $x_1, x_2, \ldots$ occur
11.11	$\mathrm{p}(x)$, $\mathrm{P}(X = x)$	probability function of the discrete random variable X
11.12	$\mathrm{p}_1, \mathrm{p}_2, \ldots$	probabilities of the values $x_1, x_2, \ldots$ of the discrete random variable X
11.13	$\mathrm{E}(X)$	expectation of the random variable X
11.14	$\mathrm{Var}(X)$	variance of the random variable X
11.15	$\sim$	has the distribution
11.16	$\mathrm{B}(n, p)$	binomial distribution with parameters n and p, where n is the number of trials and p is the probability of success in a trial
11.17	q	$q = 1 - p$ for binomial distribution
11.18	$\mathrm{N}\left(\mu, \sigma^2\right)$	Normal distribution with mean μ and variance σ^2
11.19	$Z \sim \mathrm{N}\left(0,1\right)$	standard Normal distribution
11.20	ϕ	probability density function of the standardised Normal variable with distribution $\mathrm{N}\left(0,1\right)$
11.21	Φ	corresponding cumulative distribution function
11.22	μ	population mean
11.23	σ^2	population variance
11.24	σ	population standard deviation
11.25	$\bar{x}$	sample mean
11.26	s^2	sample variance
11.27	s	sample standard deviation
11.28	H_0	null hypothesis
11.29	H_1	alternative hypothesis
11.30	r	product moment correlation coefficient for a sample
11.31	ρ	product moment correlation coefficient for a population

12	Mechanics	
12.1	kg	kilogram(s)
12.2	m	metre(s)
12.3	km	kilometre(s)
12.4	m/s, m s^{-1}	metre(s) per second (velocity)
12.5	m/s^2, m s^{-2}	metre(s) per second per second (acceleration)
12.6	F	force or resultant force
12.7	N	newton
12.8	Nm	newton metre (moment of force)
12.9	t	time
12.10	s	displacement
12.11	u	initial velocity
12.12	v	velocity or final velocity
12.13	a	acceleration
12.14	g	acceleration due to gravity
12.15	μ	coefficient of friction

A2 Appendix 2

Mathematical formulae and identities

Students must use the following formulae and identities for AS and A-level Mathematics and Further Mathematics, without these formulae and identities being provided, either in these forms or in equivalent forms. These formulae and identities may only be provided where they are the starting point for a proof or as a result to be proved.

Pure mathematics	
Quadratic equations	$ax^2 + bx + c = 0$ has roots $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Laws of indices	$a^x a^y \equiv a^{x+y}$ $a^x \div a^y \equiv a^{x-y}$ $(a^x)^y \equiv a^{xy}$
Laws of logarithms	$x = a^n \Leftrightarrow n = \log_a x$ for $a > 0$ and $x > 0$ $\log_a x + \log_a y \equiv \log_a (xy)$ $\log_a x - \log_a y \equiv \log_a \left(\dfrac{x}{y}\right)$ $k \log_a x \equiv \log_a (x^k)$
Coordinate geometry	A straight line, gradient m passing through (x_1, y_1) has equation $y - y_1 = m(x - x_1)$ Straight lines with gradients m_1 and m_2 are perpendicular when $m_1 m_2 = -1$
Sequences	General term of an arithmetic progression: $u_n = a + (n-1)d$ General term of a geometric progression: $u_n = ar^{n-1}$

Trigonometry	In the triangle *ABC*: sine rule: $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$ cosine rule: $a^2=b^2+c^2-2bc\cos A$ area: $\frac{1}{2}ab\sin C$ $\cos^2 A+\sin^2 A\equiv 1$ $\sec^2 A\equiv 1+\tan^2 A$ $\operatorname{cosec}^2 A\equiv 1+\cot^2 A$ $\sin 2A\equiv 2\sin A\cos A$ $\cos 2A\equiv \cos^2 A-\sin^2 A$ $\tan 2A\equiv \frac{2\tan A}{1-\tan^2 A}$
Mensuration	Circumference (C) and area (A) of a circle, radius r and diameter d. $C=2\pi r=\pi d$ $A=\pi r^2$ Pythagoras' Theorem: In any right-angled triangle, where a, b and c are the lengths of the sides and c is the hypotenuse: $c^2=a^2+b^2$ Area of a trapezium: $\frac{1}{2}(a+b)h$ where a and b are the lengths of the parallel sides and h is their perpendicular separation Volume of a prism $=$ area of cross section $\times$ length For a circle or radius r, where an angle at the centre of θ radians subtends an arc of length s and encloses an associated sector of area A: $s=r\,\theta$ $A=\frac{1}{2}r^2\theta$

Calculus and differential equations	Differentiation

Differentiation

Function	Derivative
x^n	nx^{n-1}
$\sin kx$	$k\cos kx$
$\cos kx$	$-k\sin kx$
e^{kx}	ke^{kx}
$\ln x$	$\frac{1}{x}$
$f(x)+g(x)$	$f'(x)+g'(x)$
$f(x)g(x)$	$f'(x)g(x)+f(x)g'(x)$
$f(g(x))$	$f'(g(x))g'(x)$

Integration

Function	Derivative
x^n	$\frac{1}{n+1}x^{n+1}+c,\ n\neq -1$
$\cos kx$	$\frac{1}{k}\sin kx+c$
$\sin kx$	$-\frac{1}{k}\cos kx+c$
e^{kx}	$\frac{1}{k}e^{kx}+c$
$\frac{1}{x}$	$\ln\lvert x\rvert+c,\ x\neq 0$
$f'(x)+g'(x)$	$f(x)+g(x)+c$
$f'(g(x))g'(x)$	$f(g(x))+c$

Area under a curve $=\int_a^b y\,dx\ (y\geq 0)$

Vectors	$\lvert x\mathbf{i} + y\mathbf{j} + z\mathbf{k}\rvert = \sqrt{(x^2 + y^2 + z^2)}$

Mechanics	
Forces and equilibrium	Weight = mass x g Friction: $F \le \mu R$ Newton's second law in the form: $F = ma$
Kinematics	For motion in a straight line with variable acceleration: $v = \dfrac{\mathrm{d}r}{\mathrm{d}t}$ $a = \dfrac{\mathrm{d}v}{\mathrm{d}t} = \dfrac{\mathrm{d}^2 r}{\mathrm{d}t^2}$ $r = \int v \, \mathrm{d}t$ $v = \int a \, \mathrm{d}t$

Statistics	
The mean of a set of data	$\overline{x} = \dfrac{\sum x}{n} = \dfrac{\sum fx}{\sum f}$
The standard Normal variable	$Z = \dfrac{X - \mu}{\sigma}$ where $X \sim \mathrm{N}\left(\mu, \sigma^2\right)$

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