Subject	A-Level Mathematics	Exam Board	AQA	Course Code	7357

Overview

The revision list for the Mocks in March is attached. It is a mock on all of Pure Mathematics. The guidance document for Pure Mathematics is attached. It may seem like a huge amount from looking at the pages but each page gives a key concept and an example of potential questions.

Use this to tick off what you know and don't know.

You also have the formula booklet here (also will be available on Microsoft Teams)

https://filestore.aqa.org.uk/resources/mathematics/AQA-AS-A-MATHS-FORMULAE.PDF

A Proof

A1 **Understand and use the structure of mathematical proof, proceeding from given assumptions through a series of logical steps to a conclusion; use methods of proof, including proof by deduction, proof by exhaustion.**

Disproof by counter example.

Proof by contradiction (including proof of the irrationality of $\sqrt{2}$ and the infinity of primes, and application to unfamiliar proofs).

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- set out a clear proof with the correct use of symbols, such as $=, \Rightarrow, \Leftarrow, \Leftrightarrow, \equiv, \therefore, \because$
- understand that considering examples can be useful in looking for structure, but this does not constitute a proof.

Notes

- At A-level 25% (20% at AS) of the assessment material must come from Assessment Objective 2 (reason, interpret and communicate mathematically). A focus on clear reasoning, mathematical argument and proof using precise mathematical language and notation should underpin the teaching of this specification. Students should become familiar with the mathematical notation found in Appendix A of the specification.
- It will not be essential to use any particular notation when writing answers to exam questions, but some questions could assess understanding of this notation.
- Students should understand the sets of numbers $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$

Examples

1 Prove that 113 is a prime number.

2 Prove that for any positive whole number, n, the value of $n^3 - n$ is always a multiple of 3.

3 "For any positive whole number, n, the value of $2n^2 + 11$ is a prime number."

Find a value of n that disproves this statement.

Only assessed at A-level

Examples

1 Assuming $\sqrt{2}$ is a rational number we can write $\sqrt{2} = \frac{a}{b}$, where a and b are positive whole numbers with no common factors.

(a) Show that a must be even.

(b) Show that b must be even.

(c) Using parts (a) and (b), explain why there is a contradiction and state what conclusion can be made about $\sqrt{2}$ as a result.

B Algebra and functions

B1 Understand and use the laws of indices for all rational exponents.

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- understand and use the following laws:

$$x^a \times x^b = x^{a+b} \qquad x^a \div x^b = x^{a-b} \qquad \left(x^a\right)^b = x^{ab}$$

$$x^{-a} = \frac{1}{x^a} \qquad x^{\frac{a}{b}} = \sqrt[b]{x^a} = \left(\sqrt[b]{x}\right)^a$$

- apply these laws when solving problems in other contexts, for example simplification of expressions before integrating/differentiating, solving equations or transforming graphs.

Examples

1 (a) Write down the values of p, q and r, given that:

(i) $a^2 = \left(a^3\right)^p$

(ii) $\dfrac{1}{\left(b^2\right)^3} = \left(b^3\right)^q$

(iii) $\sqrt{c^3} = \left(\dfrac{1}{c^2}\right)^r$

(b) Find the value of x for which

$$\frac{\left(a^3\right)^x}{a^{\frac{1}{2}}} = \frac{1}{a^2}$$

2 Find $\int\left(x+1+\frac{4}{x^2}\right)\mathrm{d}x$

3 A curve has equation $y=\frac{1}{3x^2}+4x$

Find $\frac{\mathrm{d}y}{\mathrm{d}x}$

4 Find $\int\left(\sqrt[3]{x^4}-1\right)^2\mathrm{d}x$

B2	Use and manipulate surds, including rationalising the denominator.

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- demonstrate they understand how to manipulate surds and rationalise denominators
- answer algebraic questions.

Notes

- Questions requiring simplification of surds should usually be done with a calculator, but students are expected to know the process of rationalising a surd denominator and to be able to show this step.

Examples

1 Show that $\dfrac{\sqrt{25k}-\sqrt{9k}}{\sqrt{k}}$ where k is an integer, is also an integer.

2 Express $\dfrac{3+\sqrt{a}}{2-\sqrt{a}}$, where a is a positive integer, in the form $m\sqrt{a}+n$

3 A rectangle has length $\left(9+5\sqrt{3}\right)$cm and area $\left(a+7\sqrt{3}\right)$cm^2

The width of the rectangle is $\left(m-2\sqrt{3}\right)$, where m and n are integers.

Find the value of a.

4 Show that $\dfrac{5-\sqrt{12}}{4+\sqrt{3}}$ can be written in the form $p+q\sqrt{3}$, where $p,q\in\mathbb{Z}$. Fully justify your answer.

B3

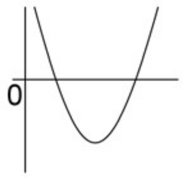
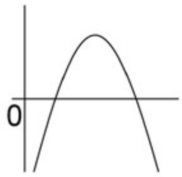
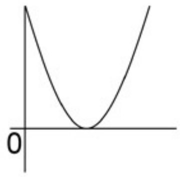
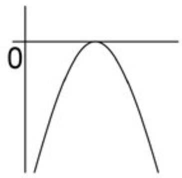
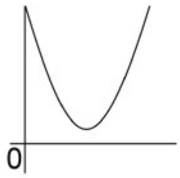
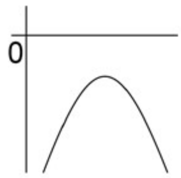
Work with quadratic functions and their graphs; the discriminant of a quadratic function, including the conditions for real and repeated roots; completing the square; solution of quadratic equations including solving quadratic equations in a function of the unknown.

Assessed at AS and A-level

Teaching guidance

Students should:

- be able to sketch graphs of quadratics, ie of $y = ax^2 + bx + c$
- be able to identify features of the graph such as points where the graph crosses the axes, lines of symmetry or the vertex of the graph
- know and use the following:

$b^2 - 4ac > 0$	Distinct real roots		
$b^2 - 4ac = 0$	Equal roots		
$b^2 - 4ac < 0$	No real roots		

Note: a quadratic described as having real roots will be such that $b^2 - 4ac \geq 0$

- be able to complete the square and use the resulting expression to make deductions, such as the maximum/minimum value of a quadratic or the number of roots

  Note: We expect students to use a calculator to solve quadratic equations and to find the coordinates of the vertex. There is no need for substitution in the quadratic formula or completing the square to justify solutions. We expect an understanding of quadratic functions, but the routine solution of equations is not in itself part of this understanding.

Teaching guidance continued

- be able to solve quadratic equations in a function of the unknown, where the function may be, for example, trigonometric or exponential.

Note: quadratic equations may arise from problems set in a variety of contexts taken from mechanics and statistics.

Examples

1 Express $3x^2 + 8x + 19$ in the form $a(x + p)^2 + q$, where a, p and q are integers.

2 Find the values of k for which the equation $x^2 - 2(k + 1)x + 2k^2 - 7 = 0$ has equal roots.

3 Show that $x = 0$ is a solution of $9^x - 3^{x+1} + 2 = 0$ and find the other solution, giving it in the form $\frac{\ln a}{\ln b}$ where a and b are integers.

4 Given that $$\frac{3 + \sin^2\theta}{\cos\theta - 2} = 3\cos\theta$$

show that $$\cos\theta = -\frac{1}{2}$$

5 Show that $$3^{2x} - 3^{x+1} - 4 = 0$$

has exactly one solution, giving this solution in an exact form. Fully justify your answer.

B4

Solve simultaneous equations in two variables by elimination and by substitution, including one linear and one quadratic equation.

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- understand the relationship between the algebraic solution of simultaneous equations and the points of intersection of the corresponding graphs
- in the case of one linear and one quadratic equation, recognise the geometrical significance of the discriminant of the resulting quadratic
- solve a pair of linear simultaneous equations using a calculator.

Notes

- Simultaneous equations could arise from problems set on a variety of topics including mechanics and statistics.
- In situations where simultaneous linear equations have to be solved to give numerical solutions, we expect students to do so with a calculator. No working is required. However, there could still be situations where coefficients are given as exact values, or as parameters, when algebraic methods will need to be used.

Examples

1 The straight line L has equation $y = 3x - 1$

The curve C has equation $y = (x + 3)(x - 1)$

(a) Sketch on the same axes the line L and the curve C, showing the values of the intercepts on the x-axis and y-axis.

(b) Show that the x-coordinates of the points of intersection of L and C satisfy the equation $x^2 - x - 2 = 0$

(c) Hence find the coordinates of the points of intersection of L and C

2 The curve C has equation $y = k(x^2 + 3)$, where k is a constant.

The line L has equation $y = 2x + 2$

Show that the x-coordinates of any points of intersection of the curve C with the line L satisfy the equation $kx^2 - 2x + 3k - 2 = 0$

3 A circle has equation

$$x^2 + y^2 - 10y + 20 = 0$$

A line has equation

$$y = 2x$$

(a) Show that the x-coordinate of any point of intersection of the line and the circle satisfies the equation $x^2 - 4x + 4 = 0$

(b) Hence, show that the line is a tangent to the circle and find the coordinates of the point where it touches the circle

4 The first term of an arithmetic series is 1
The common difference of the series is 6

(a) Find the 10th term of the series.

(b) The sum of the first n terms of the series is 7400

(i) Show that $3n^2 - 2n - 7400 = 0$

(ii) Find the value of n

Note: part (b) features content that is A-level only, and would not be asked at AS.

B5

Solve linear and quadratic inequalities in a single variable and interpret such inequalities graphically, including inequalities with brackets and fractions.

Express solutions through correct use of 'and' and 'or', or through set notation.

Represent linear and quadratic inequalities such as $y > x + 1$ and $y > ax^2 + bx + c$ graphically.

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- give the range of values which satisfy more than one inequality
- illustrate regions on sketched graphs, defined by inequalities
- define algebraically inequalities that are given graphically.

Notes

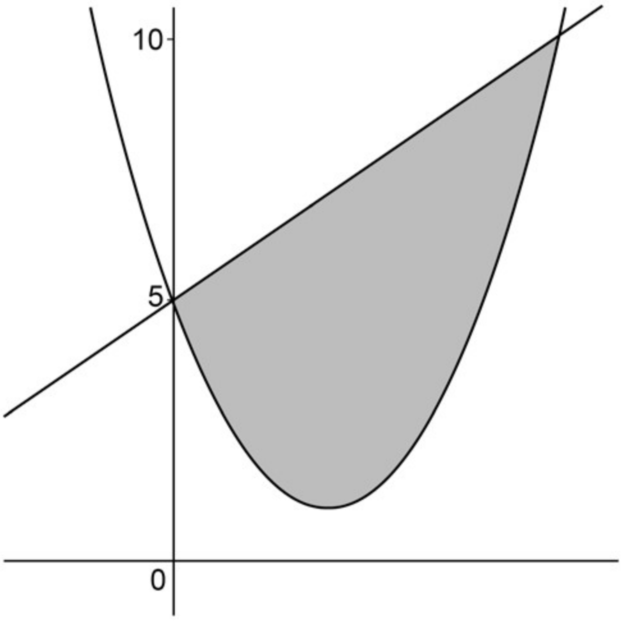
- Dotted/dashed lines or curves will be used to indicate strict inequalities.
- Overarching theme 1.3 is of particular relevance here. Students are required to demonstrate an understanding of and use the notation (language and symbols) associated with set theory (as set out in Appendix A of the specification). Students may be required to apply this notation to the solutions of inequalities.
- There are many ways of representing solution sets using set notation. Typically the variable used is the same as that used in the question, but any letter could be used. We would advise using x or y whenever possible.
- Students are expected to understand notation such as:
    - $\{x : x < 2\}$ or $(-\infty, 2)$
    - $\{x : x \leq -1\} \cup \{x : x \geq 2\}$ or $(-\infty, -1] \cup [2, \infty)$
- There is no need to state $x \in \mathbb{R}$, because we assume we are using real numbers.
- If a question requires an answer to be written in set notation we would accept any mathematically correct notation; we would accept the word 'or' instead of '$\cup$' Our intention is always to reward correct mathematics.
- Questions will always say how a required region should be indicated eg shade and label the region *R*.

Examples

1 Find the values of k which satisfy the inequality $3k^2 - 2k - 1 < 0$

Note: This question does not ask for the solution to be given in any particular form. Set notation would not be required.

2 The diagram shows the graphs of $y = x + 5$ and $y = x^2 - 4x + 5$



State which pair of inequalities defines the shaded region.

Circle your answer.

$y < x + 5$	$y \leq x + 5$	$y \leq x + 5$	$y \geq x + 5$
and	or	and	or
$y < x^2 - 4x + 5$	$y > x^2 - 4x + 5$	$y \geq x^2 - 4x + 5$	$y < x^2 - 4x + 5$

3 Find the values of x which satisfy both $x^2 + 2x > 8$ and $3(2x + 1) \leq 15$

Give your answer using set notation.

B6	**Manipulate polynomials algebraically, including expanding brackets and collecting like terms, factorisation and simple algebraic division; use of the factor theorem.** Simplify rational expressions including by factorising and cancelling, and algebraic division (by linear expressions only).

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- manipulate polynomials, which may be embedded in questions focused on other topics
- understand factorisation and division applied to a quadratic or a cubic polynomial divided by a linear term of the form $(x + a)$ where a is an integer.

Notes

- Any correct method will be accepted, eg by inspection, by equating coefficients or by formal division.
- The greatest level of difficulty is exemplified by $x^3 - 5x^2 + 7x - 3$, ie a cubic always with a factor $(x + a)$, where a is a small integer and including the cases of three distinct linear factors, repeated linear factors or a quadratic factor which cannot be factorised.
- The use of a calculator to find roots of a cubic polynomial allows students to identify linear factors using the factor theorem, although this isn't a required technique because such factors might not be of the form $(x + a)$. It would be an acceptable technique in an exam.

Examples

1 Find $\int (2x+1)(x^2 - x + 2)\,\mathrm{d}x$

2 The polynomial $\mathrm{p}(x)$ is given by $\mathrm{p}(x) = x^3 + 7x^2 + 7x - 15$

(a) Prove that $x + 3$ is a factor of $\mathrm{p}(x)$

(b) Express $\mathrm{p}(x)$ as the product of three linear factors.

3 The polynomial $p(x)$ is given by $p(x) = x^3 + x - 10$

(a) Use the factor theorem to show that $x - 2$ is a factor of $p(x)$

(b) Express $p(x)$ in the form $(x - 2)(x^2 + ax + b)$, where a and b are constants.

Only assessed at A-level

Teaching guidance

Students should be able to:

- understand the factor theorem where the divisor is of the form $(ax + b)$
- simplify rational expressions
- carry out algebraic division where the divisor is of the form $(ax + b)$.

Notes

- Any correct method will be accepted, eg by inspection, by equating coefficients or by formal division.

Examples

1 Express $\dfrac{3x^3 + 8x^2 - 3x - 5}{3x - 1}$ in the form $ax^2 + bx + \dfrac{c}{3x - 1}$, where a, b and c are integers.

2 $f(x) = 4x^3 - 7x - 3$

(a) Use the factor theorem to show that $2x + 1$ is a factor of $f(x)$

(b) Simplify $\dfrac{4x^3 - 7x - 3}{2x^2 + 3x + 1}$

B7

Understand and use graphs of functions; sketch curves defined by simple equations including polynomials, the modulus of a linear function, $\boldsymbol{y} = \frac{a}{x}$ **and** $\boldsymbol{y} = \frac{a}{x^2}$ **(including their vertical and horizontal asymptotes); interpret algebraic solution of equations graphically; use intersection points of graphs to solve equations.**

Understand and use proportional relationships and their graphs.

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- understand, use and sketch straight-line graphs (including vertical and horizontal)
- understand and use polynomials up to cubic (including sketching curves)
- understand and use cubic polynomials with at least one linear factor.
- distinguish between the various possibilities for graphs of cubic polynomials indicating where graphs meet coordinate axes
- understand and use graphs of the functions $y = \frac{a}{x}$ and $y = \frac{a}{x^2}$,
  as well as simple transformations of these graphs (including sketching curves).
- use the following:

Proportionality ($\propto$)	Equation
y is proportional to x	$y = kx$
y is proportional to x^n	$y = kx^n$
y is inversely proportional to x^n	$y = \frac{k}{x^n}$

Notes

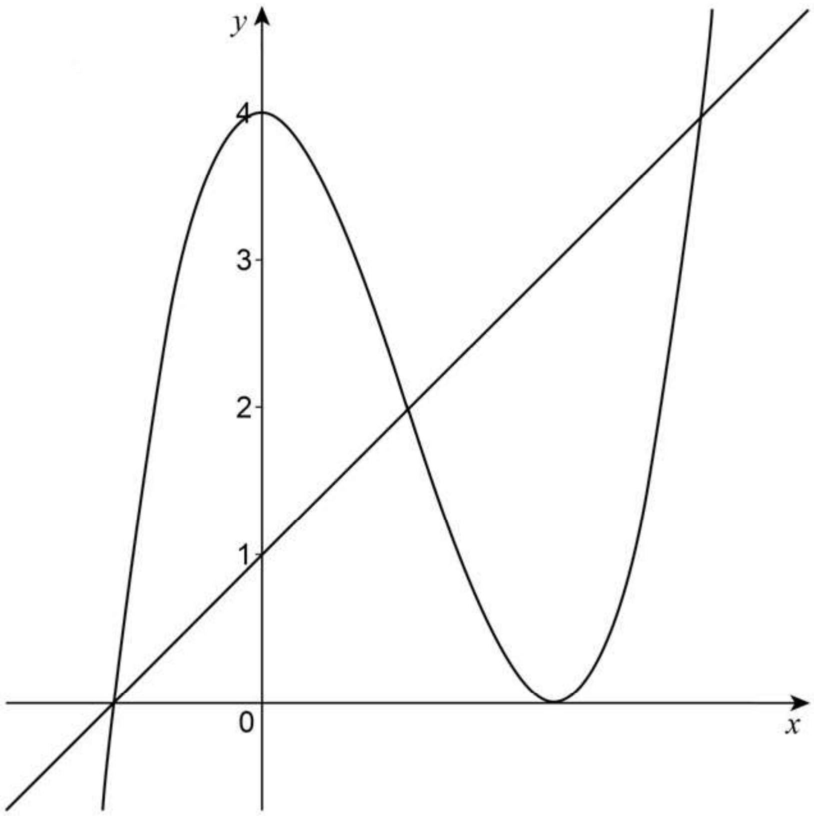
- Sketches should show roots and the y-intercept, provided these are easy to determine.
- Asymptotes should be drawn as dashed lines.

Examples

1 (a) Sketch the graph of $y = \frac{1}{x+2}$, clearly labelling the vertical asymptote.

(b) State the transformation which maps the graph of $y = \frac{1}{x+2}$ onto the graph of $y = \frac{1}{x}$

2 The graphs of $y = x^3 - ax^2 + b$ and $y = cx + d$ are shown on the diagram.



One of the points of intersection of the two graphs is (3, 4)

Find the values of a, b, c and d.

3 P is inversely proportional to the square of Q

$0 < Q < 10$

(a) Sketch the graph of P against Q

(b) The value of P increases by 1.3%

Find the percentage change in the value of Q

Only assessed at A-level

Teaching guidance

Students should:

- know and be able to use:

$$|x| = \begin{cases} x & \text{for } x \geq 0 \\ -x & \text{for } x < 0 \end{cases}$$

- understand and be able to use the graph of $y = |x|$ and combinations of simple transformations of this graph, points of intersection and solutions of equations and inequalities.

Notes

- Functions used will therefore be of the form $y = m|x| + c$ or $y = |mx + c|$

Examples

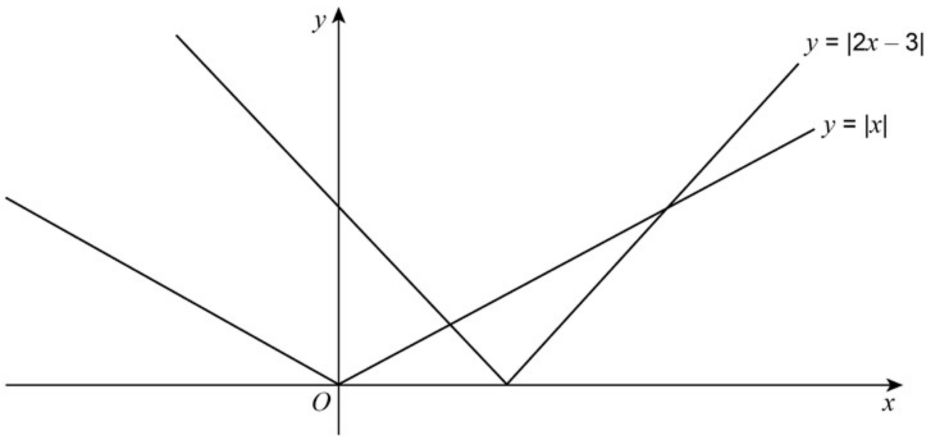
1 (a) Sketch the graph of $y = 4 - |2x|$

(b) Solve $4 - |2x| > x$

2 (a) Sketch the graph of $y = |8 - 2x|$

(b) Solve $|8 - 2x| > 4$

3 The diagram below shows the graphs of $y = |2x - 3|$ and $y = |x|$



(a) Find the exact coordinates of the points of intersection of the graphs of $y = |2x - 3|$ and $y = |x|$

(b) Hence, or otherwise, solve the inequality $|2x - 3| \geq |x|$

B8

Understand and use composite functions, inverse functions, and their graphs.

Only assessed at A-level

Teaching guidance

Students should:

- be able to define a function as a one-to-one or a many-to-one mapping, including the range and domain (co-domain not required)
- understand that the domain and range of a function are sets
- understand and be able to use correct language and notation to describe functions accurately
- know the conditions for the existence of the inverse of a function and the relationship between the domain and range of a function and those of its inverse
- understand that f^{-1} is the inverse of f if and only if $\mathrm{f}^{-1}\mathrm{f}(x) = x$ for all x
- recognise and be able to use notation such as:
  - $\mathrm{f}: x \mapsto 3x^3 - 2x + 4$
  - $\mathrm{f}(x) = x^2$
  - f^{-1} to indicate inverse
- understand that the graph of an inverse function can be found by reflecting in the line $y = x$
- understand the composition of functions:
  - $\mathrm{fg}(x)$ is $\mathrm{f}(\mathrm{g}(x))$
  - and know that the range of g must be a subset of the domain of f in order for fg to be defined.

Notes

- There are many ways of representing the domain and range sets using set notation. Typically the variable used is the same as that used in the question, but any letter could be used. We would advise using x or y whenever possible.
- Students are expected to understand notation such as:
  - $\{x : x < 2\}$ or $(-\infty, 2)$
  - $\{x : x \leq -1\} \cup \{x : x \geq 2\}$ or $(-\infty, -1] \cup [2, \infty)$
- There is no need to state $x \in \mathbb{R}$, because we assume we are using real numbers.
- If a question requires an answer to be written in set notation we would accept any mathematically correct notation; we would accept the word 'or' instead of '$\cup$' Our intention is always to reward correct mathematics.

Examples

1 The functions f and g are defined by

$f(x) = e^{2x} - 3$, for all real values of x

$g(x) = \dfrac{1}{3x-4}$, for all real values of x, $x \neq \dfrac{4}{3}$

(a) Find the range of f.

(b) Solve the equation $f^{-1}(x) = 0$

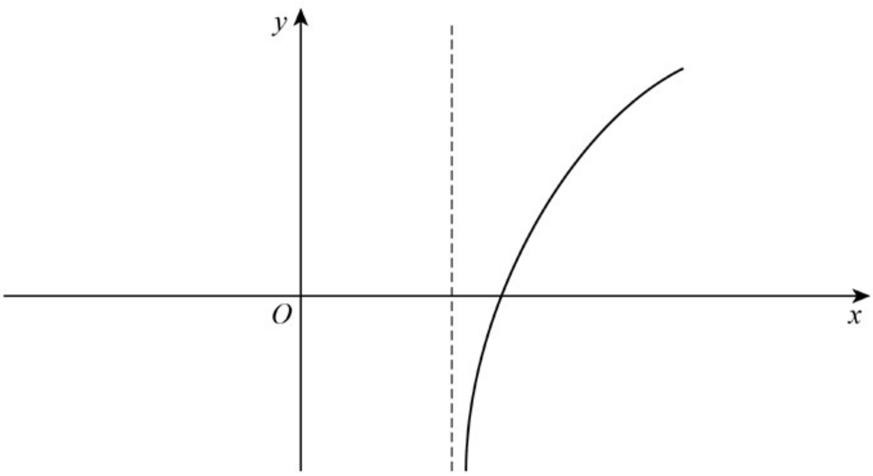
(c) (i) Find an expression for $gf(x)$

(ii) Solve the equation $gf(x) = 1$, giving your answer in exact form.

(iii) Find the exact value of x for which $gf(x)$ is undefined.

2 The curve with equation $y = f(x)$, where $f(x) = \ln(2x-3)$, is sketched below.

The domain of f is $\{x \in \mathbb{R} : x > \frac{3}{2}\}$



(a) (i) Find $f^{-1}(x)$

(ii) State the range of f^{-1}

(iii) Sketch the curve with equation $y = f^{-1}(x)$, indicating the value of the y-intercept.

(b) The function g is defined by $g(x) = e^{2x} - 4$, for all real values of x

(i) Find $gf(x)$, giving your answer in the form $(ax - b)^2 - c$, where a, b and c are integers.

(ii) Write down an expression for $fg(x)$, and hence find the exact solution of the equation $fg(x) = \ln 5$

B9	Understand the effect of simple transformations on the graph of $y = \mathrm{f}(x)$ including sketching associated graphs: $y = a\mathrm{f}(x)$, $y = \mathrm{f}(x) + a$, $y = \mathrm{f}(x + a)$ and $y = \mathrm{f}(ax)$ and combinations of these transformations.

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- describe the following:

$y = a\,\mathrm{f}(x)$	Stretch in the y-direction scale factor a
$y = \mathrm{f}(x) + a$	Translation by the vector $\begin{pmatrix} 0 \\ a \end{pmatrix}$
$y = \mathrm{f}(x + a)$	Translation by the vector $\begin{pmatrix} -a \\ 0 \end{pmatrix}$
$y = \mathrm{f}(ax)$	Stretch in the x-direction scale factor $\frac{1}{a}$

- understand that applying different transformations may result in the same function.

Examples

1 Describe a single geometrical transformation that maps the graph of $y = 3^x$

(a) onto the graph of $y = 3^{2x}$

(b) onto the graph of $y = 3^{x+1}$

2 The graph of $y = x^2 - 6x + 9$ is translated by the vector $\begin{bmatrix} -3 \\ 0 \end{bmatrix}$

Find the equation of the translated graph.

3 The graph of $y = 2^x$ is mapped by a single transformation onto the graph of $y = 2^{x-2}$

(a) Fully describe the single transformation as a translation.

(b) Fully describe the single transformation as a stretch.

Only assessed at A-level

Teaching guidance

Students should be able to:

- apply two or more transformations to a function or describe a combination of two or more transformations that result in a given function.
- understand that applying transformations in a different order may result in two different functions.

Example

1 Describe a sequence of two geometrical transformations that maps the graph of $y = \cos^{-1} x$ onto the graph of $y = 2\cos^{-1}(x-1)$

AQA

B10

Decompose rational functions into partial fractions (denominators not more complicated than squared linear terms and with no more than three terms, numerators constant or linear).

Only assessed at A-level

Teaching guidance

Students should be able to:

- use the following forms:

$$\frac{px+q}{(ax+b)(cx+d)(ex+f)} \equiv \frac{A}{(ax+b)} + \frac{B}{(cx+d)} + \frac{C}{(ex+f)}$$

$$\frac{px+q}{(ax+b)(cx+d)^2} \equiv \frac{A}{(ax+b)} + \frac{B}{(cx+d)} + \frac{C}{(cx+d)^2}$$

- understand that the fractions may need to be simplified before partial fractions are found
- understand that partial fractions may be required for integration or in a binomial approximation.

Examples

1 (a) Express $\frac{2x+3}{4x^2-1}$ in the form $\frac{A}{2x-1}+\frac{B}{2x+1}$ where A and B are integers.

(b) Evaluate $\int_4^{12} \frac{2x+3}{4x^2-1}\,\mathrm{d}x$ giving your answer in the form In q, where q is a rational number.

2 (a) Express $\dfrac{1}{(3-2x)(1-x)^2}$ in the form $\dfrac{A}{3-2x}+\dfrac{B}{1-x}+\dfrac{C}{(1-x)^2}$

(b) Solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2\sqrt{y}}{(3-2x)(1-x)^2}$$

where $y = 0$ when $x = 0$, expressing your answer in the form

$$y^p = q\ln[\mathrm{f}(x)]+\frac{x}{1-x}$$

where p and q are constants.

3 $$\mathrm{f}(x)=\frac{7x-1}{(1+3x)(3-x)}$$

(a) Express $\mathrm{f}(x)$ in the form $\dfrac{A}{3-x}+\dfrac{B}{1+3x}$ where A and B are integers.

(b) (i) Find the first three terms of the binomial expansion of $\mathrm{f}(x)$ in the form $a + bx + cx^2$, where a, b and c are rational numbers.

(ii) State why the binomial expansion cannot be expected to give a good approximation of $\mathrm{f}(0.4)$

AQA

B11

Use of functions in modelling, including consideration of limitations and refinements of the models.

Only assessed at A-level

Teaching guidance

Students should be able to:

- suggest how a model could be improved – this is a refinement
- give suggestions as to when a particular model might break down or why it is only appropriate over a particular range of values – these are limitations.

Notes

- Functions may be used in the formulation of a differential equation or arise from the solution of a differential equation.
- Modelling questions are an ideal opportunity to use calculators. Typically, exact answers will be inappropriate and thus equations can be solved using the equation solving facility of a standard scientific calculator.
- Considering the long-term behaviour of a function will often be a useful technique.

Examples

1 The total number of views, V, of a viral video clip that is released on the internet can be modelled by the formula

$$V = 150 \times 2^d$$

where d is the number of days after the video clip has been released.

(a) How many days does it take for the video to reach 1 million views?

(b) Explain why this model will eventually be inappropriate.

2 A zoologist is studying a population of 100 rodents introduced to a small island.

In order to model the size of the population, she assumes that the rate of increase of the number of rodents, $\frac{dN}{dt}$, at time t, will be proportional to the size of the population, N

This leads the zoologist to the model $N = Ae^{kt}$

(a) State the value of A

(b) Describe what happens to the population of rodents, modelled in this way, as time increases.

(c) State one criticism of this model and explain how the model could be improved.

C Coordinate geometry in the (x, y) plane

C1

Understand and use the equation of a straight line, including the forms $y - y_1 = m(x - x_1)$ and $ax + by + c = 0$; gradient conditions for two straight lines to be parallel or perpendicular.

Be able to use straight line models in a variety of contexts.

Assessed at AS and A-level

Teaching guidance

Students should:

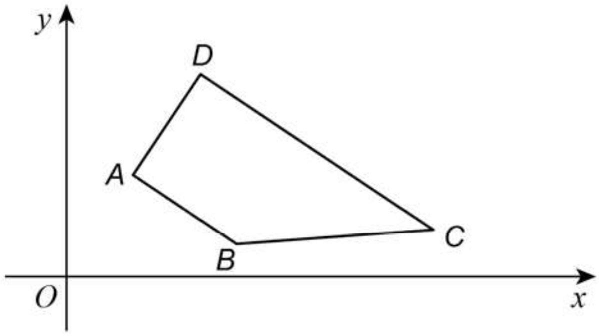
- be able to solve problems using gradients, midpoints and the distance between two points, including the form $y = mx + c$ and the forms $y = a$ and $x = b$ for horizontal and vertical lines
- know that the product of the gradients of two perpendicular lines is –1
- Understand necessary and sufficient conditions for a quadrilateral to be a square, rectangle, rhombus, parallelogram, kite or trapezium and be able to apply understanding of straight lines to these.

Notes

- In questions where the equation of a line is to be found, any correct form will be acceptable, unless specified in the question.
- However, trivial simplifications left undone in equations are likely to be penalised, eg $y - -2 = \frac{2}{4}(x-1)$ should be simplified to $y + 2 = \frac{1}{2}(x-1)$

Examples

1 The trapezium *ABCD* is shown below.



The line *AB* has equation $2x + 3y = 14$ and *DC* is parallel to *AB*.

(a) The point *D* has coordinates (3, 7)

(i) Find an equation of the line *DC*.

(ii) The angle *BAD* is a right angle.

Find an equation of the line *AD*, giving your answer in the form $mx + ny + p = 0$, where m, n and p are integers.

(b) The line *BC* has equation $5y - x = 6$. Find the coordinates of *B*.

2 The point *A* has coordinates (–1, 2) and the point *B* has coordinates (3, –5)

(a) Find an equation of the line *AB*, giving your answer in the form $px + qy = r$, where p, q and r are integers.

(b) The midpoint of *AB* is *M*. Find an equation of the line which passes through *M* and which is perpendicular to *AB*.

(c) The point *C* has coordinates $(k, 2k + 3)$

Given that the distance from *A* to *C* is $\sqrt{13}$ find the possible values of the constant k.

C2	Understand and use the coordinate geometry of the circle including using the equation of a circle in the form $(x - a)^2 + (y - b)^2 = r^2$; completing the square to find the centre and radius of a circle; use the following properties: • the angle in a semicircle is a right angle • the perpendicular from the centre to a chord bisects the chord • the radius of a circle at a given point on its circumference is perpendicular to the tangent to the circle at that point.

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- find the equation of a tangent or normal at a point
- find relevant gradients using the coordinates of appropriate points.

Note: implicit differentiation of the equation of a circle will not be required at AS, but could be expected at A-level.

Examples

1 A circle with centre *C* has equation $x^2 + y^2 - 10x + 12y + 41 = 0$

The point *A* (3, –2) lies on the circle.

(a) (i) Find the coordinates of *C*.

(ii) Show that the circle has radius $n\sqrt{5}$, where n is an integer.

(b) Find the equation of the tangent to the circle at point *A*, giving your answer in the form $x + py = q$, where p and q are integers.

(c) The point *B* lies on the tangent to the circle at *A* and the length of *BC* is 6.

Find the length of *AB*.

2 The points P (4, 3), Q (6, 7) and R (12, 4) lie on a circle, C.

(a) Show that PQ and QR are perpendicular.

(b) Find the length of PR, giving your answer as a surd.

(c) Find the equation of the circle C.

3 A circle has equation $x^2 + y^2 - 4x - 14 = 0$

(a) (i) Find the coordinates of the centre of the circle.

(ii) Find the radius of the circle in the form $p\sqrt{2}$, where p is an integer.

(b) A chord of the circle has length 8. Find the perpendicular distance from the centre of the circle to this chord.

(c) A line has equation $y = 2k - x$ where k is a constant.

(i) Show that the x-coordinate of any point of intersection of the line and the circle satisfies the equation

$$x^2 - 2(k + 1)x + 2k^2 - 7 = 0$$

(ii) Find the values of k for which the equation

$$x^2 - 2(k + 1)x + 2k^2 - 7 = 0$$

has equal roots.

(iii) Describe the geometrical relationship between the line and the circle when k takes either value found in part (c)(ii).

C3

Understand and use the parametric equations of curves and conversion between Cartesian and parametric forms.

Only assessed at A-level

Teaching guidance

Students should:

- understand that it is not always expected that the resulting Cartesian equation will be in explicit form

  Note: any correct form will be acceptable, unless stated in the question
- understand that parametric equations using trigonometric terms, often require the use of appropriate identities
- be able to answer questions requiring the gradient of a parametric curve or the tangent or normal to such a curve
- be able to find the area enclosed by a curve given in parametric form and the x-axis.

Examples

1 A curve is defined by the parametric equations

$$x = 3 - 4t \qquad y = 1 + \frac{2}{t}$$

Show that the Cartesian equation of the curve can be written as

$$(x - 3)(y - 1) + 8 = 0$$

2 A curve is given by the parametric equations

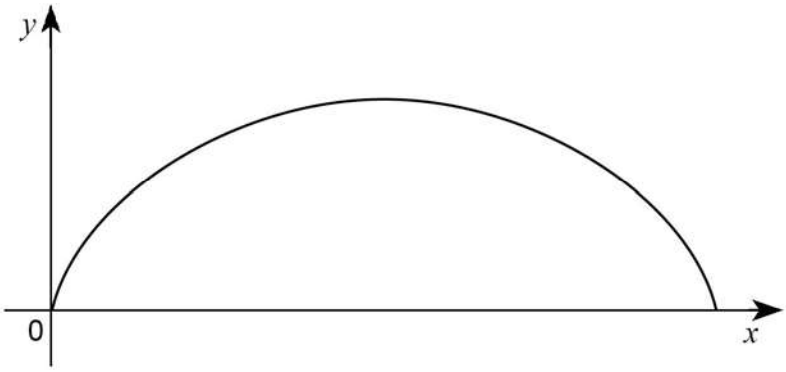
$$x = \cos\theta \qquad y = \sin 2\theta$$

Show that the Cartesian equation of the curve can be written as

$$y^2 = kx^2(1 - x^2)$$

where k is an integer.

3 The curve with parametric equations $x = 1 + \cos\theta$ and $y = \sin\theta$, $0 \le \theta \le \pi$ is shown below.



Find a Cartesian equation of the curve in the form $x^n + y^m = kx$

C4

Use parametric equations in modelling in a variety of contexts.

Only assessed at A-level

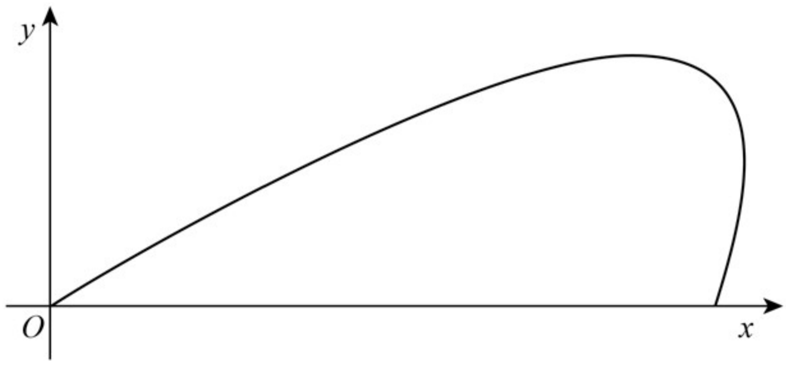
Teaching guidance

Students should be able to use parametric equations to describe the motion of a particle in the (x, y) plane, for example $x = 4t$, $y = 3t - 4.9t^2$, using the acceleration due to gravity as 9.8 m s^{-2} for a particle subject only to the force of its own weight, projected from the origin at time $t = 0$ with velocity $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$

Examples

1 A ball is thrown from an origin in a strong wind so that its horizontal and vertical position at time t is given by $x = 25t - 10t^2$ and $y = 15t - 10t^2$

The trajectory of the ball is shown in the diagram.



(a) Find the time it takes for the ball to hit the ground.

(b) Find $\dfrac{dy}{dx}$ in terms of t

(c) Use your answers to parts (a) and (b) to find the angle at which the ball hits the ground.

D Sequences and series

D1

Understand and use the binomial expansion of $(a+bx)^n$ for positive integer n; the notations $n!$ and nCr; link to binomial probabilities.

Extend to any rational n, including its use for approximation; be aware that the expansion is valid for $\left|\frac{bx}{a}\right|<1$ (Proof not required.)

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- answer questions requiring the full expansion of expressions of the form $(a+bx)^n$, where n is a small positive integer
- find the coefficients of particular powers of x (complete expansion not required)
- understand factorial notation.

Notes

- The notations $\binom{n}{r}$, ${}_nC_r$ and ${}^nC_r$ must all be recognised. Any of these may be used.
- The x in $(a+bx)^n$ may be a simple function of x, eg $\left(2-\frac{1}{x}\right)^4$

Examples

1 Show that the expansion of

$$(1+3x)^4-(1+4x)^3$$

can be written in the form

$$px^2+qx^3+rx^4$$

where p, q and r are integers.

2 Find the exact value of $\int_1^2\left[\left(2+x^{-2}\right)^3+(2-x^{-2})^3\right]\mathrm{d}x$

Only assessed at A-level

Teaching guidance

Students should be familiar with and be able to use the formulae:

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + b^n \qquad (n \in \mathbb{Z}^+)$$

where: $\binom{n}{r} = {}^n\mathrm{C}_r = \dfrac{n!}{r!(n-r)!}$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{1.2\ldots r}x^r + \ldots \qquad (|x|<1, n \in \mathbb{Q})$$

Note: the binomial expansion of $(1 + x)^n$ is valid for all real n, but it is only required for rational n in this specification.

Examples

1 (a) Find the binomial expansion of $\dfrac{1}{1+3x}$ up to the term in x^3

(b) Express $\dfrac{1+4x}{(1+x)(1+3x)}$ in partial fractions.

(c) (i) Find the binomial expansion of $\dfrac{1+4x}{(1+x)(1+3x)}$ up to the term in x^3

(ii) Find the range of values of x for which the binomial expansion of $\dfrac{1+4x}{(1+x)(1+3x)}$ is valid.

2 (a) Find the binomial expansion of $(8+6x)^{\frac{2}{3}}$ up to and including the term in x^2

(b) Use your answer from part (a) to find an estimate for $\sqrt[3]{100}$ in the form $\dfrac{a}{b}$, where a and b are integers.

Fully justify your answer.

AQA

D2	Work with sequences including those given by a formula for the n^{th} term and those generated by a simple relation of the form $x_{n+1} = \mathrm{f}(x_n)$; increasing sequences; decreasing sequences; periodic sequences.

Only assessed at A-level

Teaching guidance

Students should:

- understand and be able to use notation such as u_n
- understand that an increasing sequence will be one where $u_{n+1} > u_n$ for all n

  eg

  $u_{n+1} = 2u_n, u_1 = 5$

  $u_n = \dfrac{n}{n+1}$
- understand that a decreasing sequence will be one where $u_{n+1} < u_n$, for all n

  eg

  $u_{n+1} = 0.5u_n,\ u_1 = 5$

  $u_n = 2^{-n}$
- know that a periodic sequence repeats over a fixed interval, ie $u_{n+a} = u_n$, for all n, for a constant a and that a is the period of the sequence

  eg

  1, 2, 3, 4, 1, 2, 3, 4, …

  $u_n = \sin\left(\dfrac{\pi n}{2}\right)$
- be able to find the limit, L, of a sequence $u_{n+1} = \mathrm{f}(u_n)$ as $n \to \infty$ by setting $L = \mathrm{f}(L)$

Examples

1 A sequence is defined by

$$u_{n+1} = pu_n + q$$

where p and q are constants.

The first three terms of the sequence are given by:

$u_1 = 200$ $u_2 = 150$ $u_3 = 120$

(a) Show that p = 0.6 and find the value of q.

(b) Find the value of u_4

(c) The limit of u_n as n tends to infinity is L.

Write down an equation satisfied by L and hence find the value of L.

2 The nth term of a sequence is defined by

$$u_n = \frac{n}{n+1}$$

Prove that u_n is an increasing sequence.

3 When the fraction $\frac{1250}{999}$ is written as a decimal its digits form a periodic sequence.

What digit is in the 1000th decimal place?

D3	Understand and use sigma notation for sums of series.

Only assessed at A-level

Teaching guidance

Students should be able to:

- use $\sum$ to indicate the sum of a series, eg

  $$\sum_{r=1}^{5}(2r+1)=3+5+7+9+11$$

  $$u_n=2^{-n},\sum_{n=0}^{\infty}u_n=1+\frac{1}{2}+\frac{1}{4}+\ldots=2$$

- understand and use the notation S_n for the sum of a series
- use a calculator to find the sum of a series.

Examples

1 Find the value of

$$\sum_{r=1}^{4}\ln(2^r)$$

giving your answer in the form $p\ln 2$

2 Find the exact value of

$$\sum_{r=1}^{\infty}\frac{\sqrt{2}}{2^r}$$

3 Which of these has a different value from the other three?

Circle your answer.

$\sum_{r=1}^{k}(3r+4)$ $\quad$ $\sum_{r=2}^{k+1}(3r+1)$ $\quad$ $\sum_{r=1}^{k-1}(3r+7)$ $\quad$ $\sum_{r=5}^{k+4}(3r-8)$

D4

Understand and work with arithmetic sequences and series, including the formulae for nth term and the sum to n terms.

Only assessed at A-level

Teaching guidance

Students should know the formula for the nth term of an arithmetic sequence and be able to use the formulae for the sum of the first n terms of an arithmetic series

$u_n = a + (n-1)d$	nth term of the sequence
$S_n = \frac{n}{2}(a + l)$	Sum of first n terms using first and last term
$S_n = \frac{n}{2}(2a + (n-1)d)$	Sum of first n terms using first term and common difference

where a is the first term, d is the common difference and l is the last term of the sequence.

Notes:

Technically a series is the (infinite) sum of the terms of a sequence:

Sequence 8, 15, 22, 29, 36, …

Series 8 + 15 + 22 + 29 + 36 + …

Questions can refer to the sum of the first n terms of an arithmetic sequence or to the sum of the first n terms of an arithmetic series; both have the same meaning.

Examples

1 The first term of an arithmetic series is 1. The common difference of the series is 6

(a) Find the 10th term of the series.

(b) The sum of the first n terms of the series is 7400

(i) Show that $3n^2 - 2n - 7400 = 0$

(ii) Find the value of n

2 The 25th term of an arithmetic series is 38

The sum of the first 40 terms of the series is 1250

(a) Show that the common difference of this series is 1.5

(b) Find the number of terms in the series which are less than 100

3 This is an arithmetic series

51 + 58 + 65 + 72 + … + 1444

(a) Find the 101st term of the series.

(b) Find the sum of the last 100 terms of the series.

D5

Understand and work with geometric sequences and series including the formulae for the nth term and the sum of a finite geometric series; the sum to infinity of a convergent geometric series, including the use of $|r| < 1$; modulus notation.

Only assessed at A-level

Teaching guidance

Students should:

- know the formula for the nth term and be able to use the sum formulae:

$u_n = ar^{n-1}$	nth term of the sequence
$S_n = \frac{a(1-r^n)}{1-r}\left(=\frac{a(r^n-1)}{r-1}\right)$	sum of first n terms
$S_\infty = \frac{a}{1-r}, \lvert r\rvert<1$	sum to infinity

where a is the first term and r is the common ratio

- understand the condition for convergence of a geometric series
- understand the term 'convergent'.

Examples

1 A geometric series has first term 80 and common ratio $\frac{1}{2}$

(a) Find the sum of the first 12 terms of the series, giving your answer to two decimal places.

(b) Find the sum to infinity of the series.

2 The first three terms of a geometric sequence are x, $x + 6$ and $x + 9$

Find the common ratio of this sequence, giving your answer as a fraction in its simplest form.

3 An infinite geometric series has common ratio r. The sum to infinity of the series is five times the first term of the series.

(a) Show that $r = 0.8$

(b) Given that the first term of the series is 20, find the least value of n such that the nth term of the series is less than 1

D6 Use sequences and series in modelling.

Only assessed at A-level

Teaching guidance

Students should be able to answer questions set within a context, eg compound interest.

Examples

1 A ball is dropped from a height of 1 metre above the ground.

Each time it hits the ground it bounces to a height of $\frac{3}{4}$ the distance it fell before the bounce.

(a) Show that the distance travelled by the ball between the first and second bounce is 1.5 metres.

(b) Find the total distance travelled by the ball after it is dropped.

2 A maths graduate begins a job with a starting salary of £26 000. She has been promised an annual pay rise of 5% of the previous year's salary.

(a) How much should she expect to earn in her third year?

(b) If she stays in the same job for seven years, how much would she earn in total over this time?

E Trigonometry

E1

Understand and use the definitions of sine, cosine and tangent for all arguments; the sine and cosine rules; the area of a triangle in the form $\frac{1}{2}ab\sin C$

Work with radian measure, including use for arc length and area of sector.

Assessed at AS and A-level

Teaching guidance

Students should:

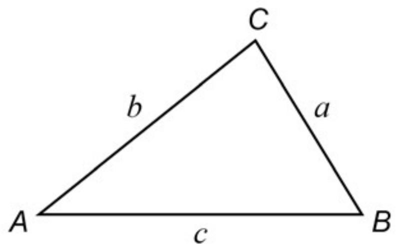
- know and be able to apply the following rules:

  In any triangle *ABC*

  • area of triangle $\frac{1}{2}ab\sin C$

  • sine rule $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

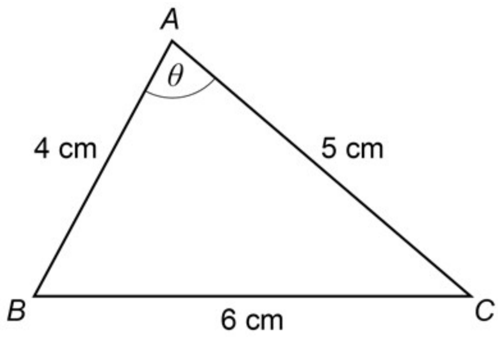
  • cosine rule $a^2 = b^2 + c^2 - 2bc\cos A$

- be aware of the ambiguous case that can arise from the use of the sine rule.

Examples

1 The triangle *ABC*, shown in the diagram, is such that *BC* = 6 cm, *AC* = 5 cm and *AB* = 4 cm. The angle *BAC* is θ.



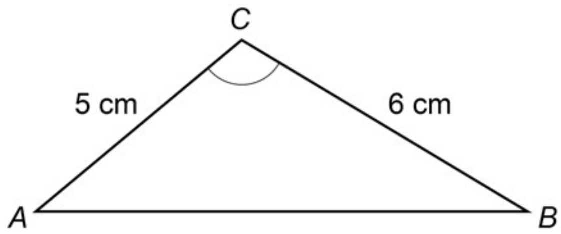
(a) Show that $\cos\theta = \frac{1}{8}$

(b) Find the exact value of the area of the triangle *ABC*.

2 Angle θ is such that $\sin\theta = \frac{1}{7}$ and $0 < \theta < 90°$

Find the exact value of $\tan\theta$

3 The diagram shows a triangle *ABC*.



The lengths of the sides *AC* and *BC* are 5 cm and 6 cm respectively.

The area of triangle *ABC* is 12.5 cm^2, and angle *ACB* is obtuse.

Find the length of *AB*, giving your answer to two significant figures.

Only assessed at A-level

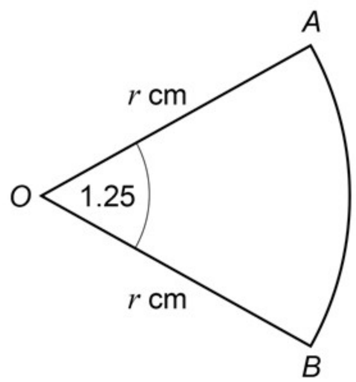
Teaching guidance

Students should:

- understand and be able to use radian measure.
- know that 2π radians $= 360°$
- know and be able to use $l = r\theta$, $A = \frac{1}{2}r^2\theta$

Examples

1 The diagram shows a sector of *OAB* of a circle with centre *O* and radius r cm.



The angle *AOB* is 1.25 radians. The perimeter of the sector is 39 cm.

(a) Show that $r = 12$

(b) Calculate the area of the sector *OAB*.

E2

Understand and use the standard small angle approximations of sine, cosine and tangent $\sin\theta \approx \theta$, $\cos\theta \approx 1-\dfrac{\theta^2}{2}$ and $\tan\theta \approx \theta$ where θ is in radians.

Only assessed at A-level

Teaching guidance

Students should be able to:

- understand and use the standard small angle approximations of sine, cosine and tangent when differentiating sine or cosine from first principles
- use standard small angle approximations to deduce approximations for other functions.

Notes

- Students should be aware that a reasonable validity for the small angle approximations is for angles less than 0.5 radians. The range of validity can be explored in greater depth, but does not need to be known for this specification.

Examples

1 (a) Show, using a small angle approximation, that $\sec x \approx \dfrac{2}{2-x^2}$

(b) Hence, find the first two terms of the binomial expansion for $\sec x$

(c) Using your binomial expansion find an approximate value for $\sec 0.1$, giving your answer to 5 decimal places.

2 (a) Using a compound angle identity write down an expression for $\sin(x+h)$

(b) Using small angle approximations for $\sin(h)$ and $\cos(h)$, and your answer to part (a), find

$$\lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}$$

E3

Understand and use the sine, cosine and tangent functions; their graphs, symmetries and periodicity.

Know and use exact values of sin and cos for $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \pi$ and multiples thereof, and exact values of tan for $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \pi$ and multiples thereof.

Assessed at AS and A-level

Teaching guidance

Students should be able to:

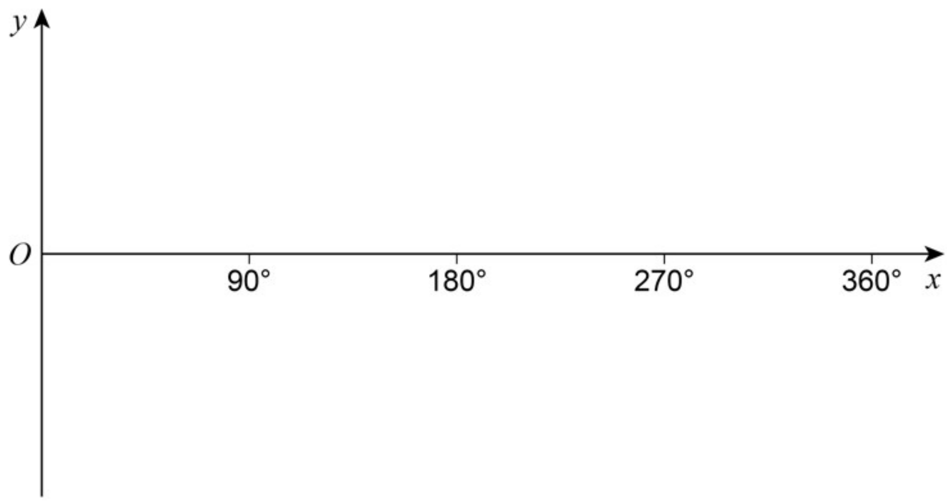
- understand and use vertical asymptotes of a tangent graph
- carry out simple transformations (as given in section B9) of the graphs of the sine, cosine and tangent functions.

At A-level combinations of transformations may be used.

Note: radians will not be required at AS.

Examples

1 (a) On the axes given below, sketch the graph of $y = \tan x$, for $0° \leq x \leq 360°$



(b) Solve the equation $\tan x = -1$, giving all the values of x in the interval $0° \leq x \leq 360°$

2 Find the two solutions of the equation $\sin x = \sin\sqrt{41}$ for $0° \le x \le 360°$

Give your answers in exact form.

Teaching guidance

Only assessed at A-level

Students should:

- understand and be able to use radians
- understand the phrase 'exact value' and use their calculators appropriately when an exact value is required.

Examples

1 Given that $\sin a = \dfrac{1}{3}$ and $0 < a < \dfrac{\pi}{2}$, find the exact value of $\sin\left(a + \dfrac{\pi}{6}\right)$

2 (a) Sketch the graph of $y = \cos x$ in the interval $0 \le x \le 2\pi$

(b) State the values of the intercepts with the coordinate axes.

E4	Understand and use the definitions of secant, cosecant and cotangent and of arcsin, arccos and arctan; their relationships to sine, cosine and tangent; understanding their graphs; their ranges and domains.

Only assessed at A-level

Teaching guidance

Students should:

- know and be able to use the following functions and their graphs:

Secant	**Cosecant**	**Cotangent**
$\sec x \equiv \dfrac{1}{\cos x}$	$\operatorname{cosec} x \equiv \dfrac{1}{\sin x}$	$\cot x \equiv \dfrac{1}{\tan x} \equiv \dfrac{\cos x}{\sin x}$

- be aware of both notations for inverse functions, eg $\arcsin x$ or $\sin^{-1} x$;
- know the domains and ranges of these functions, in particular

$$-\frac{\pi}{2} \le \sin^{-1} x \le \frac{\pi}{2}$$

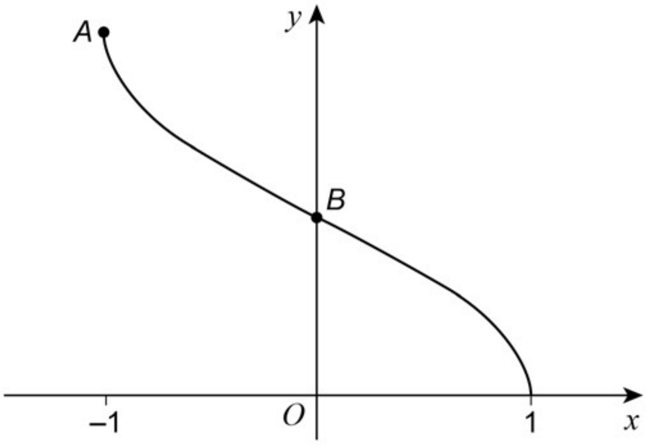
$$0 \le \cos^{-1} x \le \pi$$

$$-\frac{\pi}{2} < \tan^{-1} x < \frac{\pi}{2}$$

- understand how to sketch the graphs of the inverse trig functions by reflecting relevant sections of the trigonometric graphs in the line $y = x$
- be able to apply simple transformations (as defined in section B9) and combinations of these to graphs of all these functions.

Examples

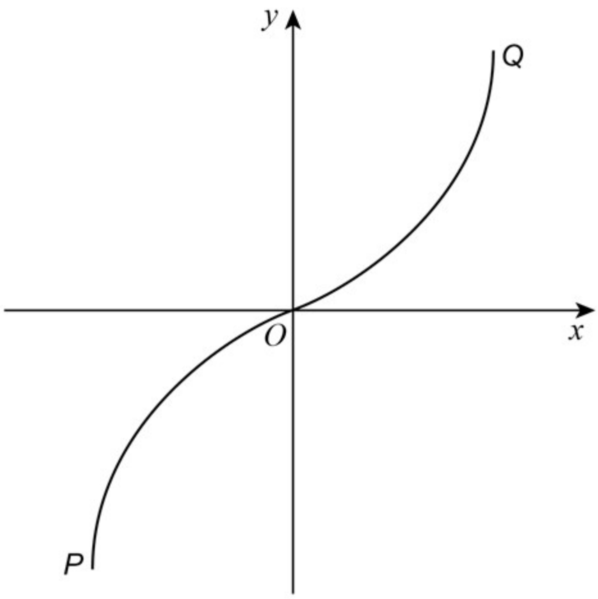
1 The diagram shows the curve $y = \cos^{-1}x$ for $-1 \le x \le 1$



Write down the exact coordinates of points *A* and *B*.

2 Sketch the curve with equation $y = \text{cosec } x$ for $0 < x < \pi$

3 (a) The sketch shows the graph of $y = \sin^{-1}x$

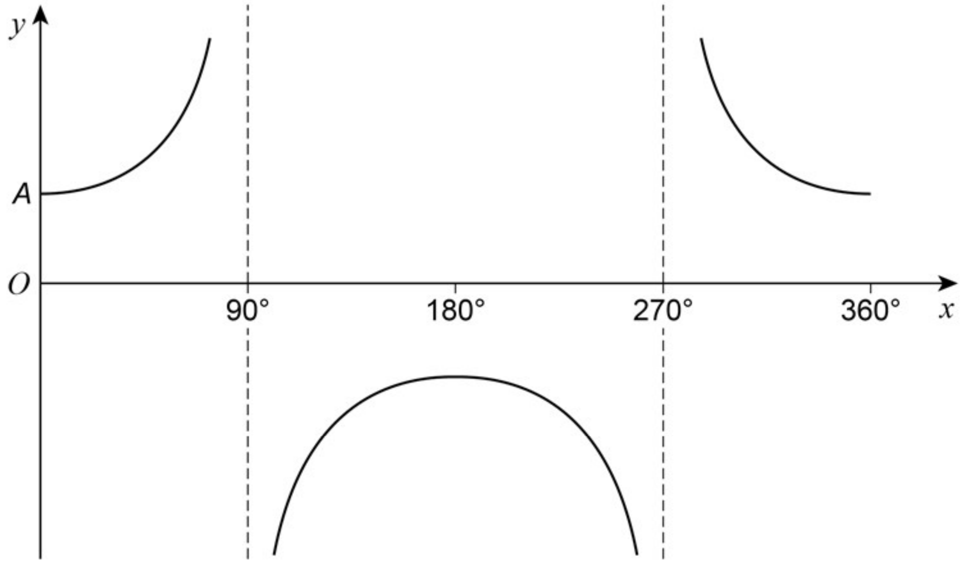


Write down the coordinates of the points *P* and *Q*, the end points of the graph.

(b) Sketch the graph of $y = \sin^{-1}(x - 1)$

4 Solve the equation sec $x = 5$, giving all the values of x in the interval $0 \le x \le 2\pi$ to two decimal places.

5 (a) The diagram shows the graph of $y = \sec x$ for $0° \leq x \leq 360°$



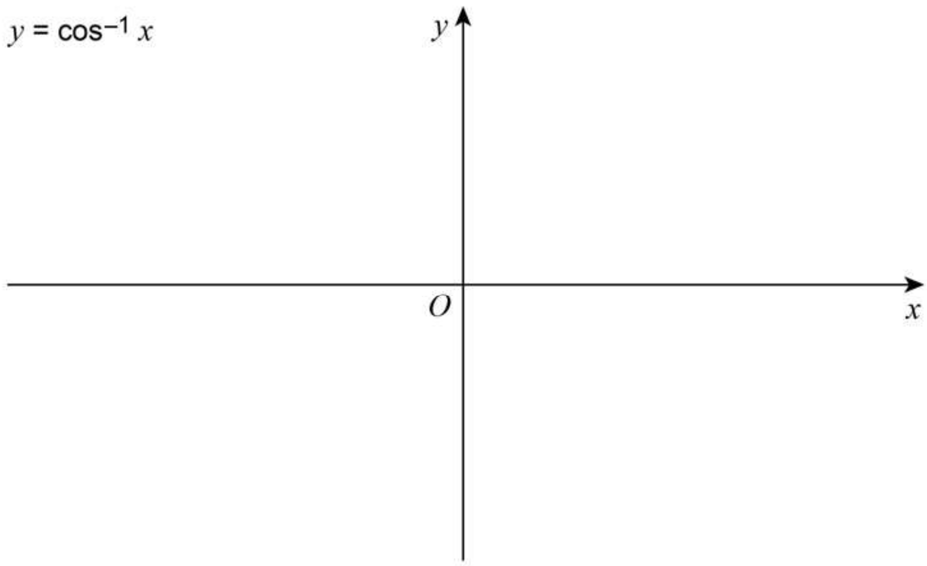
(i) The curve meets the y-axis at A. State the y-coordinate of A.

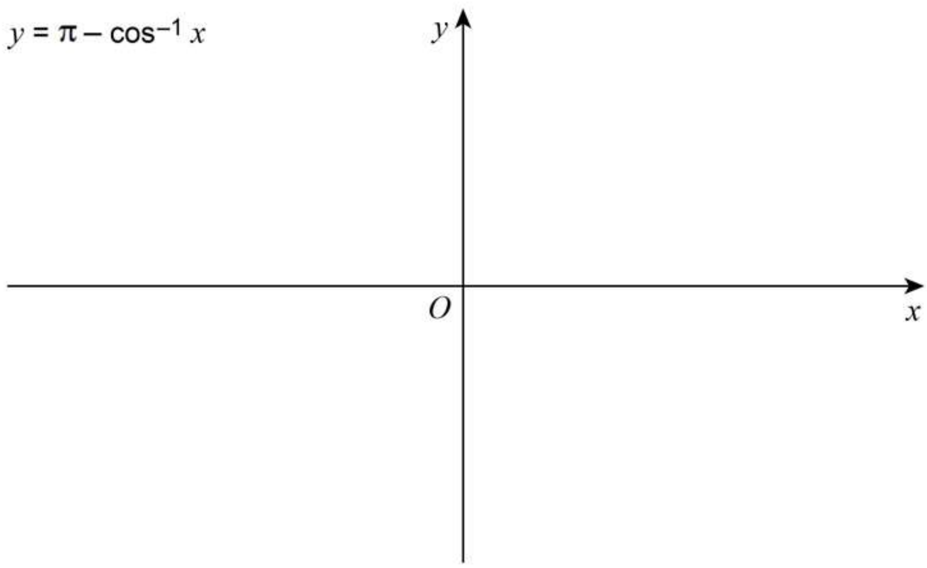
(ii) Sketch the graph of $y = \sec(2x) + 2$ for $0° \leq x \leq 360°$

(b) Solve the equation $\sec x = 2$ giving all the values of x in the interval $0° \leq x \leq 360°$

6 (a) Sketch the graph of $y = \cos^{-1} x$, where y is in radians on the first set of axes below. State the coordinates of the end points of the graph.

(b) Sketch the graph of $y = \pi - \cos^{-1} x$, where y is in radians on the second axes below. State the coordinates of the end points of the graph.





AQA

E5	Understand and use $\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$. Understand and use $\sin^2\theta + \cos^2\theta \equiv 1$; $\sec^2\theta \equiv 1 + \tan^2\theta$ and $\operatorname{cosec}^2\theta \equiv 1 + \cot^2\theta$

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- use $\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$ to solve equations or find exact values
- use $\sin^2\theta + \cos^2\theta \equiv 1$ to solve equations or find exact values.

Examples

1 (a) Given that $6\tan\theta\sin\theta = 5$, show that $6\cos^2\theta + 5\cos\theta - 6 = 0$

(b) Hence solve the equation $6\tan(3x)\sin(3x) = 5$, giving all values of x to the nearest degree in the interval $0° \le x \le 180°$

2 $(\tan\theta + 1)(\tan^2\theta - 3) = 0$

(a) Find the possible values of $\tan\theta$

(b) Hence solve the equation $(\tan\theta + 1)(\sin^2\theta - 3\cos^2\theta) = 0$, giving all solutions for θ, in degrees, in the interval $0° \le \theta \le 180°$

3 Given that $\sin\theta = \frac{2}{7}$ and θ is obtuse, find:

(a) the exact value of $\cos\theta$

(b) the exact value of $\tan\theta$

Only assessed at A-level

Teaching guidance

Students should be able to:

- use $\sec^2\theta \equiv 1+\tan^2\theta$ and $\text{cosec}^2\theta \equiv 1+\cot^2\theta$ to solve equations or find exact values
- use identities to perform integration, eg $\int \tan^2 x\,\mathrm{d}x$
- prove identities.

Examples

1 (a) Solve

$$\tan^2\theta = 3(3-\sec\theta)$$

giving all solutions to the nearest 0.1° in the interval $0° < \theta < 360°$

(b) Hence solve

$$\tan^2(4x-10°) = 3[3-\sec(4x-10°)]$$

giving all solutions to the nearest 0.1° in the interval $0° < x < 90°$

AQA

E6	Understand and use double angle formulae; use of formulae for $\sin(A \pm B)$, $\cos(A \pm B)$ and $\tan(A \pm B)$; understand geometrical proofs of these formulae. Understand and use expressions for $a\cos\theta + b\sin\theta$ in the equivalent forms of $r\cos(\theta \pm \alpha)$ or $r\sin(\theta \pm \alpha)$

Only assessed at A-level

Teaching guidance

Students should be able to:

- understand and use the addition formulae:

  $\sin(A \pm B) = \sin(A)\cos(B) \pm \cos(A)\sin(B)$

  $\cos(A \pm B) = \cos(A)\cos(B) \mp \sin(A)\sin(B)$

  $\tan(A \pm B) = \dfrac{\tan(A) \pm \tan(B)}{1 \mp \tan(A)\tan(B)}$

- understand how these formulae can be used to derive the double angle formulae:

  $\sin(2A) = 2\sin(A)\cos(A)$

  $$\begin{aligned}\cos(2A) &= \cos^2(A) - \sin^2(A)\\ &= 2\cos^2(A) - 1\\ &= 1 - 2\sin^2(A)\end{aligned}$$

- use double angle formulae to solve equations and perform integration
- use the harmonic forms $r\cos(\theta \pm \alpha)$ or $r\sin(\theta \pm \alpha)$ to solve equations or describe features of the resulting wave function, eg maximum or minimum, amplitude, etc.

Notes

- Whilst the correct formal notation is $\equiv$, it is common practice to use = when the nature of the identity is understood. Thus either $\equiv$ or $=$ will be accepted in exams.

Examples

1 Show that $\int_0^{\frac{\pi}{2}} \cos^2 x \, dx = \dfrac{\pi}{a}$, where a is an integer.

2 $3\cos\theta - 2\sin\theta \equiv R\cos(\theta + \alpha)$, where $R > 0$ and $0° < \alpha < 90°$

(a) Find the value of R.

(b) Show that $\alpha \approx 33.7°$

(c) Hence write down the maximum value of $3\cos\theta - 2\sin\theta$ and find a positive value of θ at which this maximum value occurs.

3 The polynomial $\mathrm{f}(x)$ is defined by $\mathrm{f}(x) = 4x^3 - 11x - 3$

(a) Use the factor theorem to show that $(2x + 3)$ is a factor of $\mathrm{f}(x)$

(b) Write $\mathrm{f}(x)$ in the form $(2x + 3)(ax^2 + bx + c)$

(c) Hence find all solutions of the equation $2\cos 2\theta \sin\theta + 9\sin\theta + 3 = 0$ in the interval $0° < \theta < 360°$, giving your solutions to the nearest degree.

4 (a) Show that $\cot x - \sin 2x \equiv \cot x \cos 2x$ for $0° < x < 180°$

(b) Hence, or otherwise, solve the equation $\cot x - \sin 2x = 0$ in the interval $0° < x < 180°$

5 Angle α is acute and $\cos\alpha = \frac{3}{5}$ Angle β is obtuse and $\sin\beta = \frac{1}{2}$

Show that

$$\tan(\alpha + \beta) = \frac{m\sqrt{3} - n}{n\sqrt{3} + m}$$

where m and n are integers.

E7

Solve simple trigonometric equations in a given interval, including quadratic equations in sin, cos and tan and equations involving multiples of the unknown angle.

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- understand and solve simple trigonometric equations
- answer questions that require them to give solutions in degrees.

Examples

1 Solve the equation $\sin(\theta - 30°) = 0.7$, giving your answers to the nearest 0.1° in the interval $0° \leq \theta \leq 360°$

2 Write down all solutions of the equation $\tan x = \tan 61°$ in the interval $0° \leq x \leq 360°$

3 Solve the equation $\sin 2x = \sin 48°$, giving the values of x in the interval $0° \leq x \leq 360°$

4 (a) Given that

$$\frac{\cos^2 x + 4\sin^2 x}{1 - \sin^2 x} = 7$$

show that

$$\tan^2 x = \frac{3}{2}$$

(b) Hence solve

$$\frac{\cos^2 2\theta + 4\sin^2 2\theta}{1 - \sin^2 2\theta} = 7$$

in the interval

$$0° \leq \theta \leq 180°$$

giving your values of θ to the nearest degree.

Only assessed at A-level

Teaching guidance

Students should be able to:

- understand and solve trigonometric equations using formulae from section E6, eg $\sec\left(2x+\frac{\pi}{3}\right)=\sqrt{2}$
- answer questions in radians.

Examples

1 Solve $\tan\frac{1}{2}x=3$ in the interval $0<x<4\pi$, giving your answers to three significant figures.

2 Solve the equation $\cos\theta(\sin\theta-3\cos\theta)=0$ in the interval $0<\theta<2\pi$, giving your answers to three significant figures.

3 (a) Show that the equation $2\text{cosec}^2 x=5-5\cot x$ can be written in the form

$$2\cot^2 x+5\cot x-3=0$$

(b) Hence show that $\tan x=2$ or $\tan x=-\frac{1}{3}$

(c) Hence, or otherwise, solve the equation $2\,\text{cosec}^2 x=5-5\cot x$, giving all values of x to one decimal place in the interval $-\pi\le x\le\pi$

4 (a) Solve the equation $\sec x=5$, giving all the values of x in the interval $0\le x\le 2\pi$ to two decimal places.

(b) Solve the equation $\tan^2 x=3\sec x+9$, giving all the values of x in the interval $0\le x\le 2\pi$ in radians to two decimal places.

AQA

E8	Construct proofs involving trigonometric functions and identities.

Only assessed at A-level

Teaching guidance

Students should understand and be able to use the following:

$\sin^2\theta + \cos^2\theta \equiv 1$ $\sec^2\theta \equiv 1 + \tan^2\theta$ $\text{cosec}^2\theta \equiv 1 + \cot^2\theta$

Notes

- Unless stated in the question, the above may be quoted without proof.
- Whilst the proof of an identity might typically start with the left-hand side (LHS) and deduce the right (RHS) it is equally acceptable to work from the RHS to the LHS.
- Proofs that start from both sides and meet in the middle can also be successful, but the two chains of reasoning must clearly link together.
- Proofs that transform the original identity into another, perhaps by multiplying through by a denominator, can also be successful, but often are not because of a lack of rigour.

Examples

1 Prove the identity $(\tan x + \cot x)^2 \equiv \sec^2 x + \text{cosec}^2 x$

2 Prove the identity $\dfrac{1-\sin x}{\cos x} + \dfrac{\cos x}{1-\sin x} \equiv 2\sec x$

3 Prove the identity $\dfrac{\sec^2 x}{(\sec x + 1)(\sec x - 1)} \equiv \text{cosec}^2 x$

E9

Use trigonometric functions to solve problems in context, including problems involving vectors, kinematics and forces.

Only assessed at A-level

Teaching guidance

Students should be able to:

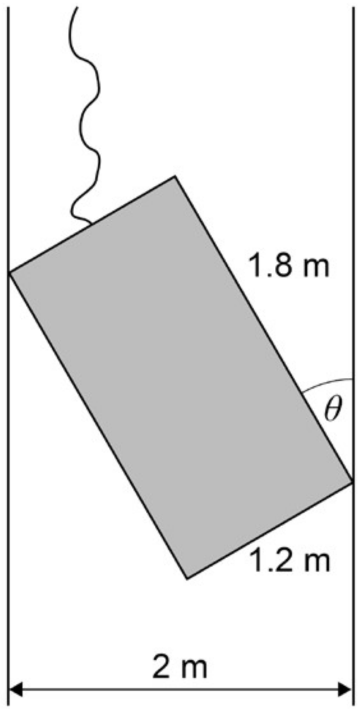
- solve problems using any of the techniques from sections E1 to E8 on their own or in combination
- select for themselves the appropriate technique for solving a problem.

Examples

1 A crane is lowering a heavy crate down a mine shaft when the crate scrapes the side of the mine shaft, twists and becomes stuck.

The mine shaft has a width of 2 metres and the crate is 1.8 metres tall by 1.2 metres wide.

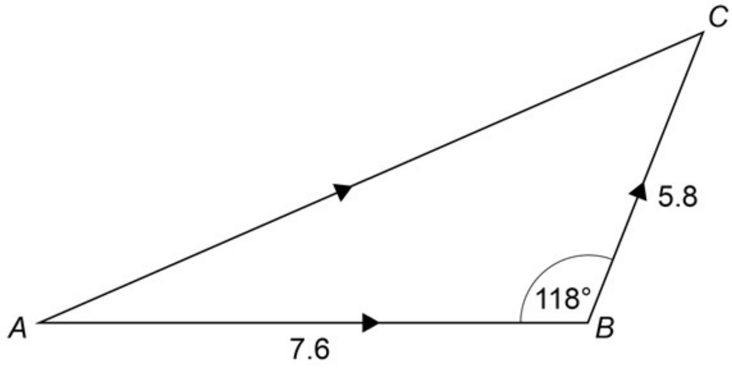
The angle between the wall of the mine shaft and the side of the crate is θ, as shown in the diagram.



(a) Show that $9\sin\theta + 6\cos\theta = 10$

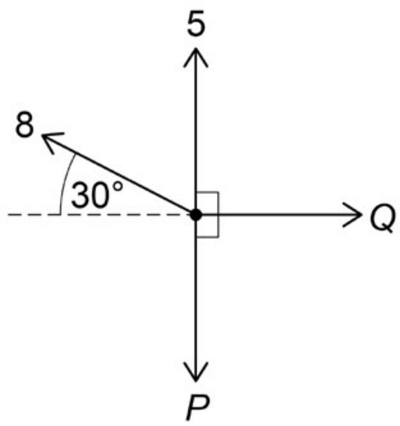
(b) Hence find the value of θ

2 Two vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$ have magnitude 7.6 units and 5.8 units respectively and the obtuse angle between the vectors is 118º, as shown in the diagram:



Find the magnitude of the vector $\overrightarrow{AC}$.

3 A particle is in equilibrium under the action of four horizontal forces of magnitudes 5 N, 8 N, P N and Q N, as shown in the diagram:



(a) Show that $P = 9$

(b) Find the value of Q.

F Exponentials and logarithms

F1

Know and use the function a^x and its graph, where a is positive.

Know and use the function e^x and its graph.

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- sketch and use simple transformations of the graph of the function a^x
- sketch and use simple transformations of the graph of the function e^x

Notes

- Simple transformations are defined in section B9.
- At A-level, students should also be able to sketch and use **a combination** of simple transformations of the functions a^x and e^x.

Examples

1 (a) Sketch the graph of $y = 3^x$, stating the coordinates of the point where the graph crosses the y-axis.

(b) Describe a single geometrical transformation that maps the graph of $y = 3^x$:

(i) onto the graph of $y = 3^{2x}$

(ii) onto the graph of $y = 3^{x+1}$

2 The curve $y = 3 \times 12^x$ is stretched by scale factor 2 parallel to the x-axis, then translated by the vector $\begin{pmatrix} 1 \\ p \end{pmatrix}$ to give the curve $y = \mathrm{f}(x)$

Given that the curve $y = \mathrm{f}(x)$ passes through the origin (0, 0), find the value of the constant p.

3 (a) Sketch the graph of $y = 9^x$, indicating the value of the intercept on the y-axis.

F2

Know that the gradient of e^{kx} is equal to ke^{kx} and hence understand why the exponential model is suitable in many applications.

Assessed at AS and A-level

Teaching guidance

Notes

At AS, students should **know** that the gradient of Ae^{kx} is proportional to the value of the function.

At AS, students are **not** expected to differentiate functions involving e^{kx}

This is an unusual situation where the gradient is expected to be known without differentiation being formally used.

Students should understand that the exponential model is suitable in many applications because, if $y = e^{kx}$, $\frac{dy}{dx} = ky$ i.e. the rate of change of y with respect to x is proportional to y.

Examples

1 Find the gradient of the curve $y = e^{2x}$ at the point where $x = 2$

Circle your answer.

$2e$ $2e^2$ e^4 $2e^4$

2 A model for the growth of a colony of bacteria is $P = 500e^{\frac{1}{8}t}$, where P is the number of bacteria after t minutes.

What is the initial rate of growth of the bacteria?

Only assessed at A-level

Teaching guidance

Students should understand and be able to use exponential functions at a more in-depth level.

Example

1 A biologist is investigating the growth of a population of a species of rodent. The biologist proposes the model

$$N = \frac{500}{1 + 9e^{-\frac{t}{8}}}$$

for the number of rodents, N, in the population t weeks after the start of the investigation.

(a) Show that the rate of growth is given by

$$\frac{N}{4000}(500 - N)$$

(b) The maximum growth occurs after T weeks. Find the value of T

AQA

F3	Know and use the definition of $\log_a x$ as the inverse of a^x, where a is positive and $x \geq 0$. Know and use the function $\ln x$ and its graph. Know and use $\ln x$ as the inverse function of e^x

Assessed at AS and A-level

Teaching guidance

Students should:

- understand and be able to use the equivalences: $y = a^x \Leftrightarrow \log_a y = x$ and $y = e^x \Leftrightarrow \ln y = x$
- know that the graph of $y = \ln x$ is a reflection in the line $y = x$ of the graph of $y = e^x$
- be able to perform simple single transformations (as defined in section B9) of the functions $y = e^x$ and $y = \ln x$
- be able to manipulate logs and exponentials within the solution to a problem.

Examples

1 (a) Sketch the graph of $y = 2\ln x$ indicating any points where the curve crosses the coordinate axes.

(b) The graph of $y = 2\ln x$ can be transformed into the graph of $y = 1 + 2\ln x$ by means of a translation. Write down the vector of the translation.

2 If $A = B^n$, which of the following is true?

Circle your answer.

$n = \log_B A$ $\quad$ $n = \log_A B$ $\quad$ $B = \log_n A$ $\quad$ $B = \log_A n$

3 Given that $\log_a b = c$, express b in terms of a and c

Only assessed at A-level

Teaching guidance

Students should be able to:

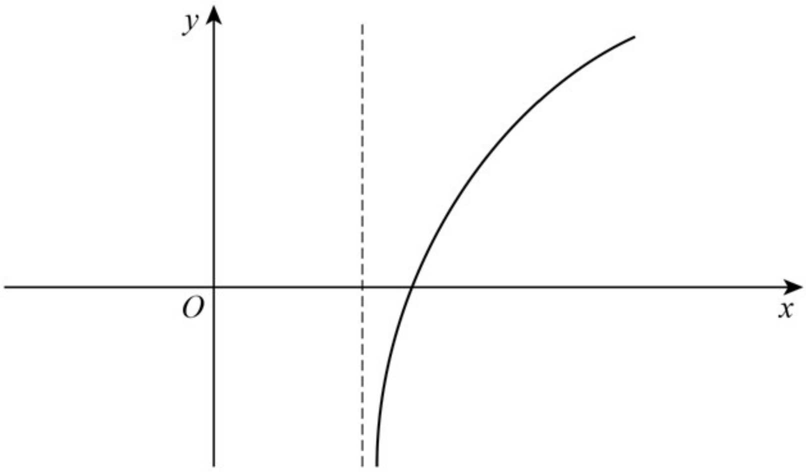
- sketch or use a combination of transformations of the functions $y = e^x$ and $y = \ln x$
- use the terms domain and range in problems using the functions $y = e^x$ and $y = \ln x$.

1 A function is defined by $f(x) = 2e^{3x} - 1$ for all real values of x

(a) Find the range of f

(b) Show that $f^{-1}(x) = \frac{1}{3}\ln\left(\frac{x+1}{2}\right)$

2 The curve with equation $y = f(x)$, where $f(x) = \ln(2x - 3)$, $x > \frac{3}{2}$, is sketched below.



The inverse of f is f^{-1}

(a) Find $f^{-1}(x)$

(b) State the range of f^{-1}

(c) Sketch the curve with equation $y = f^{-1}(x)$, indicating the coordinates of the point where the curve intersects the y-axis.

AQA

F4	Understand and use the laws of logarithms: $\log_a x + \log_a y \equiv \log_a (xy)$; $\log_a x - \log_a y \equiv \log_a \left(\frac{x}{y}\right)$; $k \log_a x \equiv \log_a x^k$ (including, for example, $k = -1$ and $k = -\frac{1}{2}$)

Assessed at AS and A-level

Teaching guidance

Students should:

- know, understand and be able to use the above laws of logarithms
- know that $\log_a a = 1$ and $\log_a 1 = 0$ for $a > 0$

Examples

1 (a) Solve $3\log_a x = \log_a 8$

(b) Show that $3\log_a 6 - \log_a 8 = \log_a 27$

2 (a) Given that $\log_a x = 2\log_a 6 - \log_a 3$, show that $x = 12$

(b) Given that $\log_a y = \log_a 5 + 7$, express y in terms of a, giving your answer in a form not involving logarithms.

Only assessed at A-level

Example

1 The curve $y = 3^x$ intersects the line $y = x + 3$ at the point where $x = \alpha$

(a) Show that α lies between 0.5 and 1.5

(b) Show that the equation $3^x = x + 3$ can be arranged into the form

$$x = \frac{\ln(x+3)}{\ln 3}$$

F5 Solve equations of the form $a^x = b$.

Assessed at AS and A-level

Teaching guidance

Students should be able to solve equations of the form $a^x = b$, including $e^x = b$

Notes

- Equations of this form may require exact answers.
- If exact answers are not required such equations may be solved using a calculator, unless instructions are given to the contrary.

Examples

1 The line $y = 85$ intersects the curve $y = 6^{3x}$ at the point *A*. Find the x-coordinate of *A*, giving your answer to three decimal places.

2 The line $y = 21$ intersects the curve $y = 3(2^x + 1)$ at the point *P*

(a) Show that the x-coordinate of *P* satisfies the equation $2^x = 6$

(b) Find the x-coordinate of *P*, giving your answer to three significant figures.

3 Given that $e^{-2x} = 3$, find the exact value of x

F6	Use logarithmic graphs to estimate parameters in relationships of the form $y = ax^n$ and $y = kb^x$, given data for x and y.

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- reduce a non-linear relationship to linear form
- plot a graph from given data, drawing a line of best fit by eye and using it to calculate the gradient and intercept to estimate for unknown constants.

Note: this is an essential skill in A-level sciences and there is an ideal opportunity here to link to real data: power laws for relationships of the form $y = ax^n$ and exponential laws for those of the form $y = kb^x$

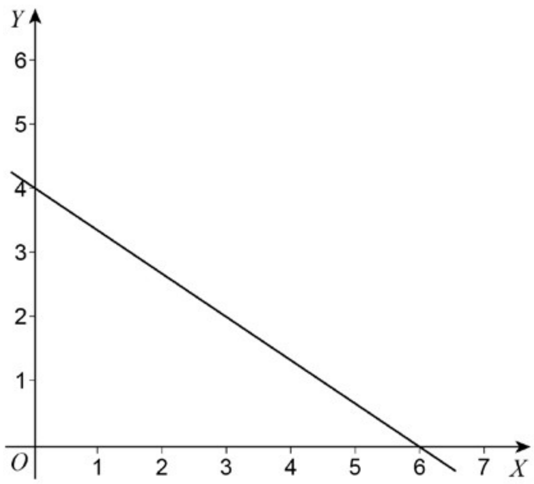
Examples

1 The variables y and x are related by an equation of the form $y = ax^n$ where a and n are constants.

Let $Y = \log_{10} y$ and $X = \log_{10} x$

(a) Show that there is a linear relationship between Y and X

(b) The graph of Y against X is shown in the diagram.



Find the value of n and the value of a

2 The variables x and y are known to be related by an equation of the form $y = ab^x$ where a and b are constants.

The following approximate values of x and y have been found.

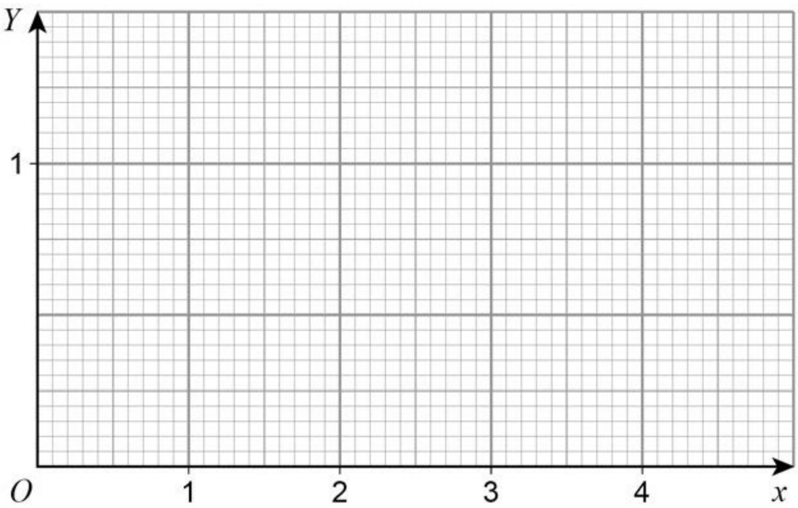
x	1	2	3	5
y	3.84	6.14	9.82	15.8

(a) Complete the table, showing values of x and Y, where $Y = \log_{10} y$. Give each value of Y to three decimal places.

x	1	2	3	4
Y	0.584			

(b) Show that, if $y = ab^x$, then x and Y must satisfy an equation of the form $Y = mx + c$

(c) Draw a graph relating x and Y



(d) Use a line of best fit on your graph to find estimates for the values of a and b

F7

Understand and use exponential growth and decay; use in modelling (examples may include the use of e in continuous compound interest, radioactive decay, drug concentration decay, exponential growth as a model for population growth); consideration of limitations and refinements of exponential models.

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- use given conditions to determine the values of unknown constant terms in equations of the forms $y = Ae^{bx} + C$ or $P = Ak^t + C$
- form and use exponential equations to make predictions.

Examples

1 On 1 January 1900, a sculpture was valued at £80

When the sculpture was sold on 1 January 1956, its value was £5000

The value, £V, of the sculpture is modelled by the formula $V = Ak^t$, where t is the time in years since 1 January 1900 and A and k are constants.

(a) Write down the value of A

(b) Show that $k = 1.07664$ to five decimal places.

(c) Use this model to:

(i) show that the value of the sculpture on 1 January 2006 will be greater than £200 000

(ii) find the year in which the value of the sculpture will first exceed £800 000

(d) Explain whether your answer to (c)(ii) is, in reality, likely to be accurate.

2 A biologist is researching the growth of a certain species of hamster. She proposes that the length, x cm, of a hamster t days after its birth is given by $x = 15 - 12e^{-\frac{t}{14}}$

(a) Use this model to find:

(i) the length of a hamster when it is born.

(ii) the length of a hamster after 14 days, giving your answer to three significant figures.

(b) (i) Show that the time for a hamster to grow to 10 cm in length is given by

$$t = 14\ln\left(\frac{a}{b}\right)$$

where a and b are integers.

(ii) Find this time to the nearest day.

3 The concentration, C mg per litre of a particular drug in a patient's bloodstream t hours after it has been administered is given by the formula $C = C_0 e^{-0.2t}$

(a) Initially a patient is given a dose of 5 mg per litre.

(i) Write down the value of C_0

(ii) Find the concentration of the drug 3 hours after it is administered.

(b) The drug becomes ineffective when the concentration drops below 2 mg per litre

How long does it take for the drug to become ineffective? Give your answer to the nearest minute.

Note: within every set of exam papers at AS, 20% of the assessment must address AO3. At A-level, 25% of assessment addresses AO3.

Examples of modelling should be introduced to students at an early stage of teaching so that they can build confidence in the use of models and in the interpretation of the outputs from mathematical models. Models will not always be given to students. Students will sometimes be required to translate a situation in context into a mathematical model. Teachers should be mindful of both OT2 and OT3 because many assessment items will be set in the context of these overarching themes, which address problem-solving and modelling.

G Differentiation

G1

Understand and use the derivative of $\mathbf{f}(x)$ as the gradient of the tangent to the graph of $y = \mathbf{f}(x)$ at a general point (x, y); the gradient of the tangent as a limit; interpretation as a rate of change; sketching the gradient function for a given curve; second derivatives; differentiation from first principles for small positive integer powers of x and for sin x and cos x

Understand and use the second derivative as the rate of change of gradient; connection to convex and concave sections of curves and points of inflection.

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- recognise and use the notations $\mathrm{f}'(x)$, $\frac{\mathrm{d}y}{\mathrm{d}x}$, $\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{f}(x))$
- use $\mathrm{f}''(x)$ or $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ to determine the nature of a stationary point. (See section G3)

Examples

1 A model helicopter takes off from a point O at time $t = 0$ and moves vertically so its height, y cm, above O after time t seconds is given by:

$$y = \frac{1}{4}t^4 - 26t^2 + 96t\ ,\ 0 \le t \le 4$$

(a) Find:

(i) $\dfrac{dy}{dt}$

(ii) $\dfrac{d^2y}{dt^2}$

(b) Verify that y has a stationary value when $t = 2$ and determine whether this stationary value is a maximum value or a minimum value.

(c) Find the rate of change of y with respect to t when $t = 1$

(d) Determine whether the height of the helicopter above O is increasing or decreasing at the instant when $t = 3$

(e) Determine whether the speed of the helicopter is increasing or decreasing at the instant when $t = 3$

2 The volume, V m^3, of water in a tank at time t seconds is given by

$$V = \frac{1}{3}t^6 - 2t^4 + 3t^2 \text{ for } t \ge 0$$

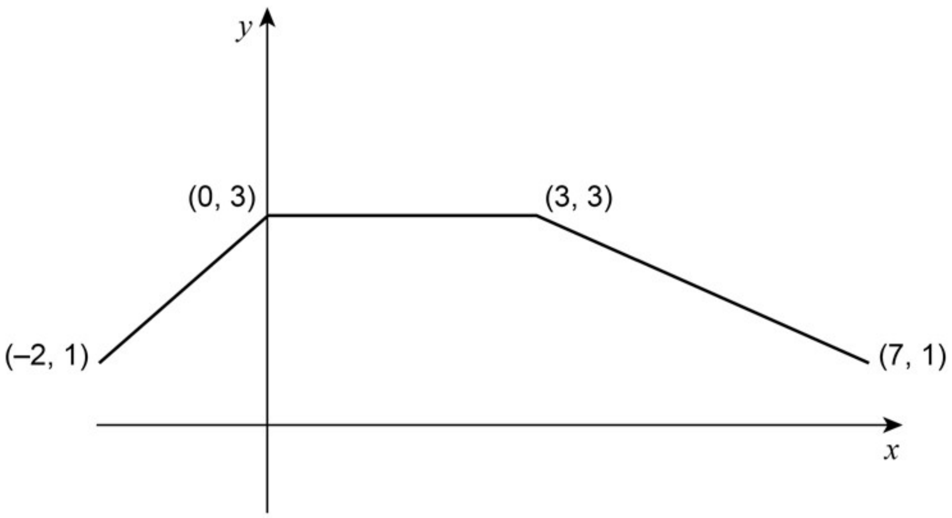
(a) Find:

(i) $\dfrac{dV}{dt}$

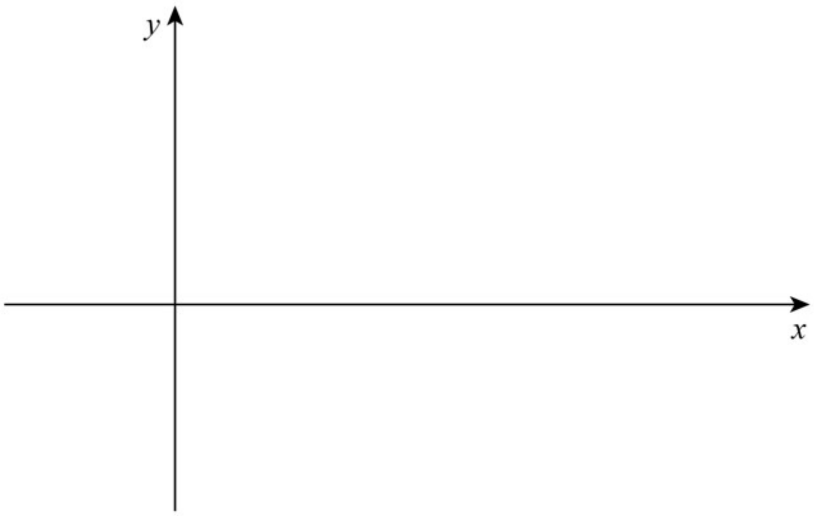
(ii) $\dfrac{d^2V}{dt^2}$

(b) Find the rate of change of the volume of water in the tank, in m^3 s^{-1}, when $t = 2$

3 The graph of $y = \mathrm{f}(x)$ is shown below.



Sketch the graph of $y = \mathrm{f}'(x)$ on the axes below.



4 A curve has equation $y = x^3 - 12x$

The point *A* on the curve has coordinates (2, –16)

The point *B* on the curve has x-coordinate $2 + h$

(a) Show that the gradient of the line *AB* is $6h + h^2$

(b) Explain how the result of part (a) can be used to show that *A* is a stationary point on the curve.

Only assessed at A-level

Teaching guidance

Students should:

- know that at points of inflection $f''(x) = 0$, but that the converse is not necessarily true.
- know that for concave and convex sections of a function over the closed interval $[a, b]$ the following holds:
  - a twice-differentiable function is concave $\Leftrightarrow f''(x) \leq 0$ for all $x \in [a, b]$
  - a twice-differentiable function is convex $\Leftrightarrow f''(x) \geq 0$ for all $x \in [a, b]$

Notes

- Students should verify the nature of a point of inflection, either by considering a change in concavity or by showing there is no change in sign for the first derivative.

Examples

1 A curve is given by the equation $f(x) = x^3 - 3x^2 + 1$

Find the range of values of x for which the curve is concave.

2 The point $A\left(\frac{\pi}{3}, \frac{1}{2}\right)$ is on the curve $y = \cos x$

The point *B* on the curve has x-coordinate $\frac{\pi}{3} + h$

(a) Show that the gradient of the line *AB* can be written as

$$\frac{\cos h - \sqrt{3}\sin h - 1}{2h}$$

(b) Using the result from part (a) and small angle approximations explain how the gradient of the curve, $y = \cos x$, at the point *A* can be found.

AQA

G2

Differentiate x^n, for rational values of n, and related constant multiples, sums and differences.

Differentiate e^{kx} and a^{kx}, sin kx, cos kx and tan kx, related sums, differences and constant multiples.

Understand and use the derivative of ln x.

Assessed at AS and A-level

Teaching guidance

Students should know and be able to use the following:

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1} \qquad \frac{d}{dx}\left(Ax^n\right) = Anx^{n-1} \qquad \frac{d}{dx}\left(Ax^n + Bx^m\right) = Anx^{n-1} + Bmx^{m-1}$$

Example

1 $x = \frac{1}{2}t^4 - 20t^2 + 66t$

Find:

(a) $\frac{dx}{dt}$

(b) $\frac{d^2x}{dt^2}$

Only assessed at A-level

Teaching guidance

Students should know and be able to use the following:

$\mathbf{f}(x)$	$\mathbf{f}'(x)$
e^{kx}	ke^{kx}
$\ln x$	$\frac{1}{x}$
$\sin kx$	$k\cos kx$
$\cos kx$	$-k\sin kx$
$\tan kx$	$k\sec^2 kx$
a^{kx}	$(k\ln a)a^{kx}$

The following is given in the formulae booklet:

$\tan x$	$\sec^2 x$

Examples

1 Find $\frac{dy}{dx}$ when $y = e^{3x} + \ln x$

2 A curve has the equation $y = e^{2x} - 10e^2 + 12x$

(a) Find $\frac{dy}{dx}$

(b) Find $\frac{d^2y}{dx^2}$

3 Find $\frac{dy}{dx}$ when $y = \tan 3x$

G3	Apply differentiation to find gradients, tangents and normals, maxima and minima and stationary points, points of inflection. Identify where functions are increasing or decreasing.

Assessed at AS and A-level

Teaching guidance

Students should:

- understand and be able to use the fact that at a stationary point, $\frac{dy}{dx} = 0$
- describe a stationary point as a (local) maximum or minimum
- know that:

  At a maximum $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$

  At a minimum $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} > 0$

  Note: the case $\frac{d^2y}{dx^2} = 0$ will not be tested at AS

- use $m_1 \times m_2 = -1$ for gradients of tangent and normal
- be able to answer questions set in the form of a practical problem where a function of a single variable has to be optimised
- be able to show that a function is increasing or decreasing, by showing $\frac{dy}{dx} > 0$ or $\frac{dy}{dx} < 0$ respectively.

Notes:

- There are different interpretations of increasing and decreasing functions. Whilst our preferred definition is that a function is increasing (decreasing) on an open interval (a, b) if $\frac{dy}{dx} > 0$ $\left(\frac{dy}{dx} < 0\right)$ then the definition that permits the derivative to equal zero is also acceptable.

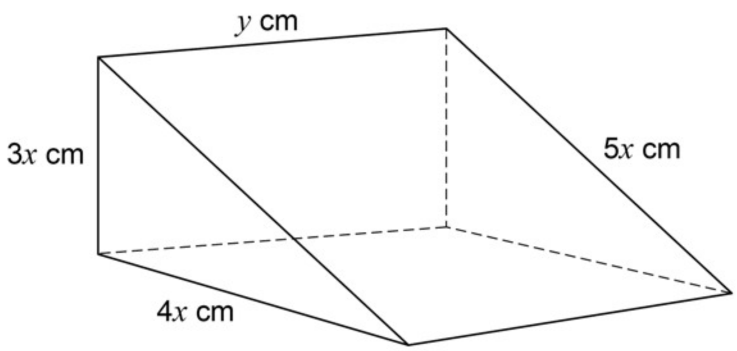
Examples

1 The curve with equation $y = x^4 - 32x + 5$ has a single stationary point, *M*

Find the coordinates of *M* and determine its nature.

Fully justify your answer.

2 The diagram shows a block of wood in the shape of a prism with a triangular cross-section. The end faces are right-angled triangles with sides of lengths $3x$ cm, $4x$ cm and $5x$ cm, as shown in the diagram.



The total surface area of the five faces is 144 cm^2.

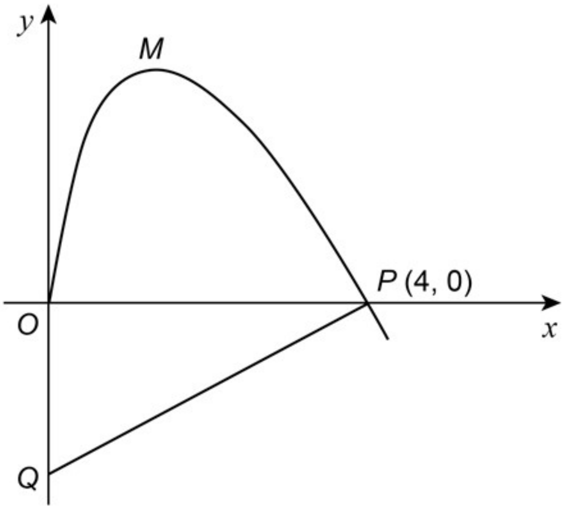
(a) Show that the volume of the block, V cm^3, is given by $V = 72x - 6x^3$

(b) Show that V has a stationary value when $x = 2$ and determine whether it is a maximum or a minimum.

Fully justify your answer.

3 A curve, drawn from the origin O, crosses the x-axis at the point $P(4, 0)$

The normal to the curve at P meets the y-axis at the point Q, as shown in the diagram.



The curve, defined for $x \geq 0$, has the equation $y = 4x^{\frac{1}{2}} - x^{\frac{3}{2}}$

(a) Find $\frac{dy}{dx}$

(b) Show that the gradient of the curve at P is -2

(d) Find the area of triangle OPQ

Only assessed at A-level

Teaching guidance

Students should:

- distinguish between stationary points and turning points
- know that a point of inflection is where a curve changes from convex to concave or vice versa and use this to test for points of inflection
- know that some points of inflection are stationary points, but that more often they are non-stationary points of inflection
- understand that $\frac{d^2y}{dx^2} = 0$ at a point of inflection, but this is not, on its own, sufficient to show the existence of a point of inflection, eg $y = x^4$ has a minimum at $x = 0$ (not a point of inflection) although $\frac{d^2y}{dx^2} = 0$ when $x = 0$
- be able to determine the nature of a stationary point when the second derivative is zero, by considering the sign of the first derivative either side of the stationary point.

Example

1 A curve has the equation $y = e^x + e^{-x}$

Show that any points of inflection of this curve satisfy $e^{2x} = -1$

2 A curve has the equation $y = e^{-2x} + 6x$

(a) Find the exact values of the coordinates of the stationary point of the curve.

(b) Determine the nature of the stationary point.

Fully justify your answer.

AQA

G4

Differentiate using the product rule, the quotient rule and the chain rule, including problems involving connected rates of change and inverse functions.

Only assessed at A-Level

Teaching guidance

Students should:

- know and be able to use the following rules:

Product rule	$\frac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{f}(x)\mathrm{g}(x)\right]=\mathrm{f}'(x)\mathrm{g}(x)+\mathrm{f}(x)\mathrm{g}'(x)$
Quotient rule (this is given in the formulae booklet)	$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\mathrm{f}(x)}{\mathrm{g}(x)}\right]=\frac{\mathrm{f}'(x)\mathrm{g}(x)-\mathrm{f}(x)\mathrm{g}'(x)}{\left[\mathrm{g}(x)\right]^2}$
Chain rule	If $y=\mathrm{f}(u)$ and $u=\mathrm{g}(x)$, so that $y=\mathrm{f}(\mathrm{g}(x))$ then $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}u}\times\frac{\mathrm{d}u}{\mathrm{d}x}$
Inverses	$\frac{\mathrm{d}x}{\mathrm{d}y}=\frac{1}{\frac{\mathrm{d}y}{\mathrm{d}x}}$

- be able to use product, quotient or chain rules for differentiating $\tan x$, $\sec x$, $\operatorname{cosec} x$ and $\cot x$

Examples

1 (a) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ when:

(i) $y=\left(4x^2+3x+2\right)^{10}$

(ii) $y=x^2\tan x$

(b) (i) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ when $x=2y^3+\ln y$

(ii) Hence find an equation of the tangent to the curve $x=2y^3+\ln y$ at the point (2, 1)

2 (a) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ when:

(i) $y = (2x^2 - 5x + 1)^{20}$

(ii) $y = x\cos x$

(b) Given that

$$y = \frac{x^3}{x-2}$$

show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{kx^2(x-3)}{(x-2)^2}$$

where k is a positive integer.

3 A bucket is being used to catch water dripping from a leaking classroom ceiling, at a constant rate of 0.5 cm^3 per minute.

The volume, V cm^3, of water in the bucket is given by

$$V = \frac{\pi h}{3}\left(h^2 + 3h + 300\right)$$

Where h is the depth of water in the bucket.

Find the rate of change of h in cm per minute when the depth of water in the bucket is 10 cm.

G5

Differentiate simple functions and relations defined implicitly or parametrically, for first derivative only.

Only assessed at A-Level

Teaching guidance

Students should be able to:

- use the chain rule with parametric equations

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} \quad \text{or} \quad \frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$$

- use the chain rule with parametric equations to find stationary points, equations of tangents and normals, but **not** to find points of inflection **nor** to test for concavity
- differentiate implicitly, using the product rule, quotient rule and chain rule as appropriate.

Examples

1 A curve is defined by the parametric equations

$$x = 1 + 2t, \quad y = 1 - 4t^2$$

(a) Hence find $\frac{dy}{dx}$ in terms of t

(b) Find an equation of the normal to the curve at the point where $t = 1$

2 A curve is defined by the equation

$$y^2 - xy + 3x^2 - 5 = 0$$

(a) Find the y-coordinates of the two points on the curve where $x = 1$

(b) (i) Show that $\frac{dy}{dx} = \frac{y - 6x}{2y - x}$

(ii) Find the gradient of the curve at each of the points where $x = 1$

(iii) Show that, at the two stationary points on the curve, $33x^2 - 5 = 0$

3 A curve is defined by the parametric equations $x = 2\cos\theta$, $y = 3\sin 2\theta$

(a) (i) Show that

$$\frac{dy}{dx} = a\sin\theta + b\operatorname{cosec}\theta$$

where a and b are integers.

(ii) Find the gradient of the normal to the curve at the point where $\theta = \frac{\pi}{6}$

(b) Show that the Cartesian equation of the curve can be expressed as

$$y^2 = px^2(4 - x^2)$$

where p is a rational number.

4 A curve is defined by the equation $9x^2 - 6xy + 4y^2 = 3$

Find the coordinates of the two stationary points of this curve.

5 (a) A curve is defined by the equation $x^2 - y^2 = 8$

(i) Show that at any point (p, q) on the curve, where

$$q \neq 0$$

the gradient of the curve is given by

$$\frac{dy}{dx} = \frac{p}{q}$$

(ii) Show that the tangents at the points (p, q) and $(p, -q)$ intersect on the x-axis.

(b) Show that

$$x = t + \frac{2}{t},\quad y = t - \frac{2}{t}$$

are parametric equations of the curve

$$x^2 - y^2 = 8$$

AQA

G6

Construct simple differential equations in pure mathematics and in context, (contexts may include kinematics, population growth and modelling the relationship between price and demand).

Only assessed at A-level

Teaching guidance

Students should understand language associated with proportionality and rates of change.

Examples

1 A water tank has a height of 2 metres. The depth of the water in the tank is h metres at time t minutes after water begins to enter the tank. The rate at which the depth of water in the tank increases is proportional to the difference between the height of the tank and the depth of the water.

Write down a differential equation in the variables h and t and a positive constant k

2 A biologist is investigating the growth of a population of a species of rodent. The biologist proposes the model

$$N = \frac{500}{1 + 9e^{-\frac{t}{8}}}$$

for the number of rodents, N, in the population, t weeks after the start date of the investigation.

Use this model to answer the following questions.

(a) Show that the rate of growth is given by

$$\frac{N}{4000}(500 - N)$$

(b) The maximum rate of growth occurs after T weeks. Find the value of T

3 A giant snowball is melting. The snowball can be modelled as a sphere whose surface area is decreasing at a constant rate with respect to time. The surface area of the sphere is A cm^2 at time t days after it begins to melt.

Write down a differential equation in terms of the variables A and t and a constant k, where $k > 0$, to model the surface area of the melting snowball.

4 The number of fish in a lake is decreasing. After t years, there are x fish in the lake. The rate of decrease of the number of fish is proportional to the number of fish currently in the lake.

(a) Formulate a differential equation, in the variables x and t and a constant of proportionality k, where $k > 0$, to model the rate at which the number of fish in the lake is decreasing.

(b) At a certain time, there were 20 000 fish in the lake and the rate of decrease was 500 fish per year. Find the value of k

H Integration

H1 Know and use the Fundamental Theorem of Calculus.

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- understand that differentiation is the 'reverse' of integration and vice versa
- use $\int_a^b f(x)\,dx = F(b) - F(a)$

  where $\frac{d}{dx}\left(F(x)\right) = f\left(x\right)$

  Note: the maximum level of difficulty for questions at AS requires students to use an integrand, $f(x)$, where $f(x)$ is the sum of terms of the form ax^n where n is rational and $n \neq -1$
- be able to find a function given its derivative and boundary condition.

Examples

1 The curve $y = f(x)$ passes through (1, 3)

$f'(x) = (2x-3)(x+1)^2$

Find $f(x)$

2 $g(x) = \frac{d}{dx}\int_1^x \left(t^2 - 3t\right)dt$

Find $g(3)$

H2

Integrate x^n (excluding $n = -1$), and related sums, differences and constant multiples.

Integrate e^{kx}, $\frac{1}{x}$, sin kx, cos kx and related sums, differences and constant multiples.

Assessed at AS and A-level

Teaching guidance

Students should:

- know that:

$$\int x^n \mathrm{d}x = \frac{x^{n+1}}{n+1} + c\,,\ n = -1$$

  and

$$\int ax^n + bx^m \mathrm{d}x = a\int x^n \mathrm{d}x + b\int x^m \mathrm{d}x$$

- understand integration as the reverse of differentiation
- include a constant of integration when finding an indefinite integral.

Examples

1 Find $\int \left(1 - \frac{1}{x^2}\right)^3 \mathrm{d}x$

2 Find $\int \left(1 + 3x^{\frac{1}{2}} + x^{\frac{3}{2}}\right) \mathrm{d}x$

AQA

Only assessed at A-level

Teaching guidance

Students should know and be able to use the following:

$\mathrm{f}(x)$	$\int \mathrm{f}(x)\,\mathrm{d}x$
e^{kx}	$\frac{1}{k}e^{kx} + c$
$\frac{1}{x}$	$\ln x + c$
$\sin kx$	$-\frac{1}{k}\cos kx + c$
$\cos kx$	$\frac{1}{k}\sin kx + c$

Notes

- Whilst it is correct to say that $\int \frac{1}{x}\,\mathrm{d}x = \ln|x| + c$, the use of the modulus notation is not required at A-level.

Examples

1 Find $\int 2e^{4x} + \frac{1}{2x}\,\mathrm{d}x$

2 (a) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ when $y = x\ln x$

(b) Hence find $\int \ln x\,\mathrm{d}x$

3 Find $\int 4\sin 2x - 12\cos 3x\;\mathrm{d}x$

H3

Evaluate definite integrals; use a definite integral to find the area under a curve and the area between two curves.

Assessed at AS and A-level

Teaching guidance

Students should be able to:

- understand and use the fact that for a function, f, where $f(x) \geq 0$ for $a \leq x \leq b$ the area between the x-axis, the curve $y = f(x)$ and the lines $x = a$ and $x = b$ is given by

$$\text{area} = \int_a^b f(x)\,dx$$

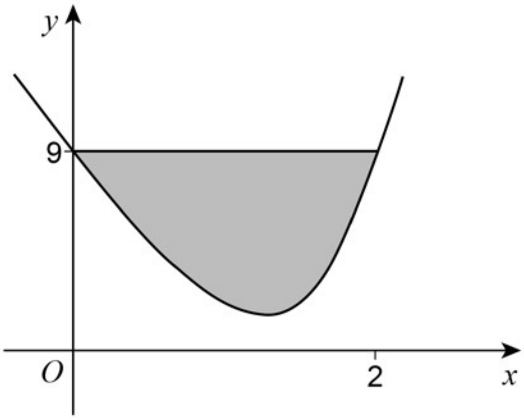
- understand that for areas lying **below** the x-axis the definite integral will give the negative of the required value
- find areas between curves and straight lines.

Notes

- Definite integrals can be found on a calculator and students are expected to do this in exams. If exact answers are required, these will usually require a non-calculator method.
- Students are **not** expected to find an area between a curve and the y-axis, by integrating an expression for x with respect to y

Examples

1 The curve with equation $y = x^4 - 8x + 9$ is sketched below.

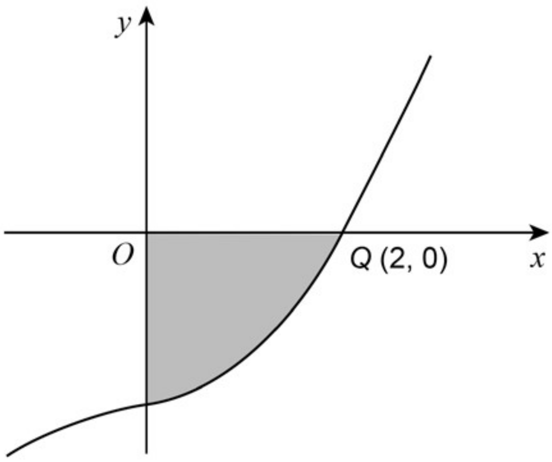


The point (2, 9) lies on the curve.

Find the area of the shaded region bounded by the curve and the line $y = 9$

Note: we would expect this question to be done on a calculator. The mathematical principles being tested in such a question are that the area under the curve can be found by integration and that the required area can be found by subtracting from the area of a rectangle.

2 The curve *C* with equation $y = x^3 + x - 10$, sketched below, crosses the x-axis at the point *Q* (2, 0)



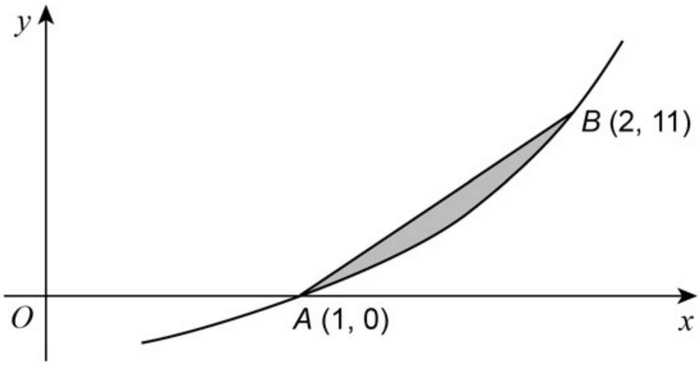
(a) Find an equation of the tangent to the curve *C* at the point *Q*

(b) Find $\int (x^3 + x - 10)\,dx$

(c) Find the area of the shaded region bounded by the curve *C* and the coordinate axes.

Note: this part can be done using a calculator.

3 The curve with equation $y = x^3 + 4x - 5$ is sketched below.



The curve cuts the x-axis at the point *A*(1, 0) and the point *B*(2, 11) lies on the curve.

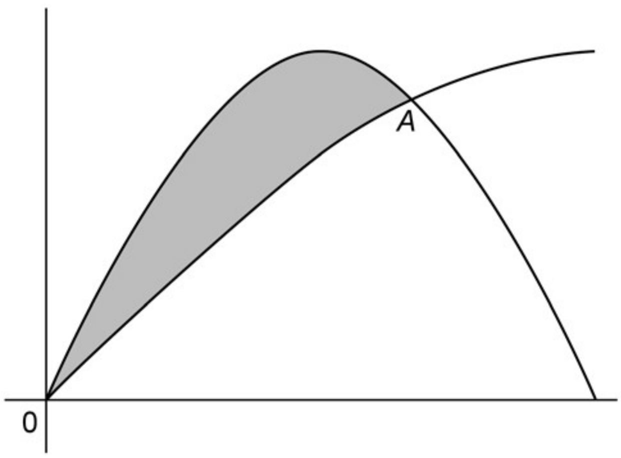
Find the area of the shaded region bounded by the curve and the line *AB*.

Note this can be considered as the area between two "curves," but the method expected is to subtract the area under the curve from the area of a triangle.

Only assessed at A-level

Example

1 The two curves, $y = \sin x$ and $y = \sin 2x$, for $0 \leq x \leq \frac{\pi}{2}$, are shown on the graph.



Find the exact area of the shaded region.

H4	Understand and use integration as the limit of a sum.

Only assessed at A-level

Teaching guidance

Students should be able to:

- understand that the area under a curve can be approximated using rectangles. The limit as the number of rectangles is increased is equal to a definite integral
- recognise and use notation such as $\lim_{n \to \infty} \sum_{i=1}^{n} y_i \delta x = \int_a^b y \, \mathrm{d}x$

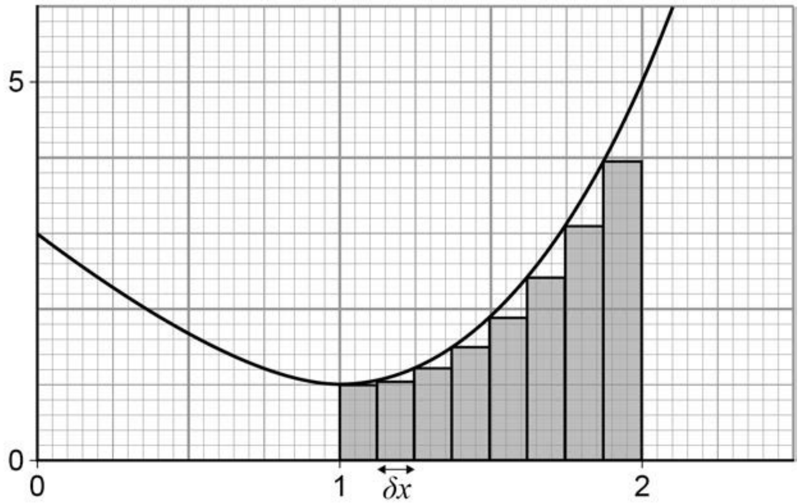
Notes: the idea here is to link to work at GCSE using the trapezium rule and to have a semi-formal understanding of the ideas of Riemann sums, although such vocabulary is not required. Students should comment on whether an approximation would give an over- or underestimate or if it cannot be decided, linking this to the ideas of increasing and decreasing functions.

The ideas here can be investigated using interactive applets such as intmath.com/integration/riemann-sums.php

This is an example of how the use of technology should permeate the course.

Example

1 The area under the graph of $y = x^3 - 3x + 3$ between the lines $x = 1$ and $x = 2$ is to be estimated by calculating the areas of the 8 rectangles, of equal width, δx, as shown in the diagram.



(a) What is the value of δx?

(b) The area of the smallest rectangle can be found using the calculation $y_1 \times \delta x$, where y_1 is the height of the rectangle.

Calculate $\sum_{i=1}^{8} y_i \delta x$

(c) A more accurate estimate of the area under the curve can be found by using more rectangles of smaller width.

Calculate $\lim_{n \to \infty} \sum_{i=1}^{n} y_i \delta x$

H5	Carry out simple cases of integration by substitution and integration by parts; understand these methods as the inverse processes of the chain and product rules respectively. (Integration by substitution includes finding a suitable substitution and is limited to cases where one substitution will lead to a function which can be integrated; integration by parts includes more than one application of the method but excludes reduction formulae.)

Only assessed at A-level

Teaching guidance

Students should be able to:

- understand the very simplest cases of substitution such as:

  $\int e^{ax+b} dx$

  $\int \sin(ax+b) dx$

  $\int \frac{1}{ax+b} dx$

Note: it is likely students will learn these as standard integrals rather than use substitution each time.

- recognise integrals of the form

  $\int f'(x) \cdot [f(x)]^n dx$

  and integrate directly or by substitution

- use $\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$

  to perform integration by parts, limited to a maximum of two consecutive applications of the method

- choose a simple substitution to perform an integral such as $\int \frac{x}{\sqrt{x+1}} dx$ (This example is not exhaustive and students should have experience of various simple substitutions.)
- integrate curves given in parametric form and find the area under a curve given in parametric form.

Examples

1 Find $\int x^2 \sin 2x \, \mathrm{d}x$

2 Use the substitution $x = \sin\theta$ to find

$$\int \frac{1}{\left(1-x^2\right)^{\frac{3}{2}}} \mathrm{d}x$$

giving your answer in terms of x

3 (a) (i) Find $\int \ln x \, \mathrm{d}x$

(ii) Find $\int (\ln x)^2 \, \mathrm{d}x$

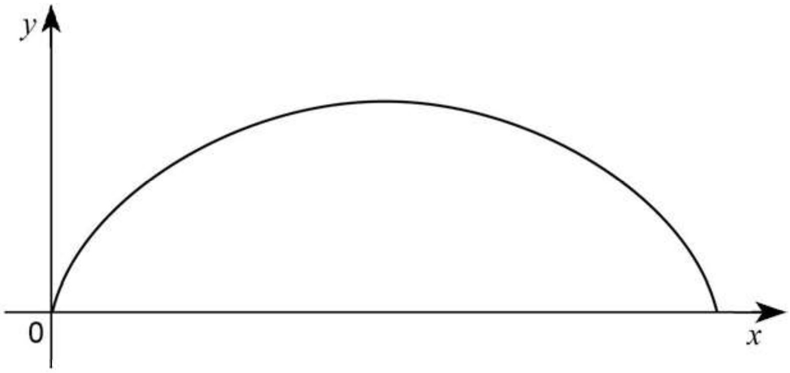
(b) Use the substitution $u = \sqrt{x}$ to find the exact value of $\int_1^4 \frac{1}{x+\sqrt{x}} \mathrm{d}x$

4 Find $\int x^2 \left(x^3 + 2\right)^7 \mathrm{d}x$

5 Find $\int \frac{x^2 + 2x}{2x^3 + 6x^2 + 1} \mathrm{d}x$

6 Find $\int \frac{\sin x}{\cos x} \mathrm{d}x$

7 The curve with parametric equations $x = 1 + \cos\theta$ and $y = \sin\theta, 0 \le \theta \le \pi$ is shown below.

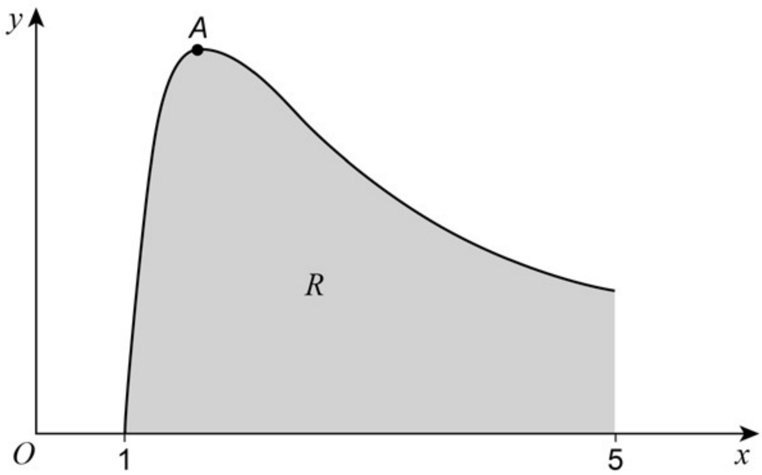


Find the area enclosed between the curve and the x-axis.

8 (a) Given that $y = x^{-2} \ln x$, show that $\frac{dy}{dx} = \frac{1 - 2\ln x}{x^3}$

(b) Using integration by parts, find $\int x^{-2} \ln x \, dx$

(c) The sketch shows the graph of $y = x^{-2} \ln x$



Find the exact value of the shaded area R.

H6

Integrate using partial fractions that are linear in the denominator.

Only assessed at A-level

Teaching guidance

Students should be able to answer questions that require the simplification of a more complicated fraction, which leads to an integrable expression or to partial fractions that can be integrated.

Examples

1 (a) Find $\int \frac{5x-6}{x(x-3)}\,\mathrm{d}x$

(b) (i) $$4x^3 + 5x - 2 \equiv (2x+1)(2x^2 + px + q) + r$$

Find the values of the constants p, q and r

(ii) Hence, find $\int \frac{4x^3+5x-2}{2x+1}\,\mathrm{d}x$

2 (a) (i) Express

$$\frac{5-8x}{(2+x)(1-3x)}$$

in the form

$$\frac{A}{2+x}+\frac{B}{1-3x}$$

where A and B are integers.

(ii) Hence show that

$$\int_{-1}^{0}\frac{5-8x}{(2+x)(1-3x)}\,\mathrm{d}x = p\ln 2$$

where p is rational.

H7

Evaluate the analytical solution of simple first order differential equations with separable variables, including finding particular solutions.

(Separation of variables may require factorisation involving a common factor.)

Only assessed at A-level

Teaching guidance

Students should be able to carry out any of the techniques of integration included in sections H1 to H6.

Examples

1 Solve the differential equation

$$\frac{dy}{dx} = 6xy^2$$

given that $y = 1$ when $x = 2$

Give your answer in the form $y = f(x)$

2 Solve the differential equation

$$\frac{dy}{dx} = \frac{1}{y}\cos\left(\frac{x}{3}\right)$$

given that $y = 1$ when $x = \frac{\pi}{2}$

Write your answer in the form $y^2 = f(x)$

3 (a) Solve the differential equation

$$\frac{dy}{dt} = y \sin t$$

to obtain y in terms of t

(b) Given that $y = 50$ when $t = \pi$, show that $y = 50e^{-(1+\cos t)}$

4 Find the general solution of the differential equation

$$\frac{dy}{dx} = xy + 2x$$

Give your answer in the form $y = f(x)$

5 Solve the differential equation

$$\frac{dy}{dx} = \frac{x\sqrt{x^2 + 3}}{e^{2y}}$$

given that $y = 0$ when $x = 1$

Give your answer in the form $y = f(x)$

H8

Interpret the solution of a differential equation in the context of solving a problem, including identifying limitations of the solution; includes links to kinematics.

Only assessed at A-level

Teaching guidance

Students should be able to solve differential equations they have set up themselves.

Examples

1 A car's value **depreciates** at a rate which is proportional to its value, £V, at time t months from when it was new.

(a) Write down a differential equation in terms of the variables V and t and a constant k, where $k > 0$, to model the value of the car.

(b) Solve your differential equation to show that $V = Ad^t$ where $d = e^{-k}$

(c) The value of the car when new was £12 499 and 36 months later its value was £7000

Find the values of A and d

2 The platform of a theme park ride oscillates vertically. For the first 75 seconds of the ride

$$\frac{dx}{dt} = \frac{t\cos\left(\frac{\pi}{4}t\right)}{32x}$$

where x metres is the height of the platform above the ground after time t seconds.

At $t = 0$, the height of the platform above the ground is 4 metres.

Find the height of the platform after 45 seconds, giving your answer to the nearest centimetre.

AQA

I Numerical methods

I1

Locate roots of $f(x) = 0$ by considering changes of sign of $f(x)$ in an interval of x on which $f(x)$ is sufficiently well-behaved.

Understand how change of sign methods can fail.

Only assessed at A-level

Teaching guidance

Students should:

- be able to answer questions where roots of equations in the form $g(x) = h(x)$ can be located by rearranging to give $f(x) = g(x) – h(x) = 0$
- know that discontinuities can cause failure but also know that the converse is not true
- clearly state that a curve is continuous when concluding that a root exists in a given interval.

Examples

1 The curve $y = 3^x$ intersects the line $y = x + 3$ at the point where $x = \alpha$

Show, using the change of sign method, that α lies between 0.5 and 1.5

2 Show, using the change of sign method, that the equation

$$x^3 – 6x + 1 = 0$$

has a root α, where

$$2 < \alpha < 3$$

3 When searching for a root to the equation $f(x) = 0$, a student correctly evaluates $f(1) = 4$ and $f(2) = –3$

Given that f is a continuous function, which one of the following could not be true?

Circle the correct answer.

There are no roots for $1 < x < 2$	There is one root for $1 < x < 2$	There are two roots for $1 < x < 2$	There are three roots for $1 < x < 2$

I2	Solve equations approximately using simple iterative methods; be able to draw associated cobweb and staircase diagrams. Solve equations using the Newton-Raphson method and other recurrence relations of the form $x_{n+1} = \mathrm{g}(x_n)$. Understand how such methods can fail.

Only assessed at A-level

Teaching guidance

Students should be able to:

- answer questions that require them to rearrange an equation into an iterative form
- draw staircase and cobweb diagrams, to illustrate the iteration, on printed graphs
- demonstrate divergence on a diagram
- understand the conditions when the Newton-Raphson method may fail.

Notes

- This is an opportunity to use technology. Interactive software, such as GeoGebra, allows you to investigate cobweb diagrams: geogebra.org/m/XvjM7Xnv
- Students must use calculators to perform iterative operations.

Examples

1 (a) The equation $e^{-x} - 2 + \sqrt{x} = 0$ has a single root, α

Show that α lies between 3 and 4

(b) Use the recurrence relation

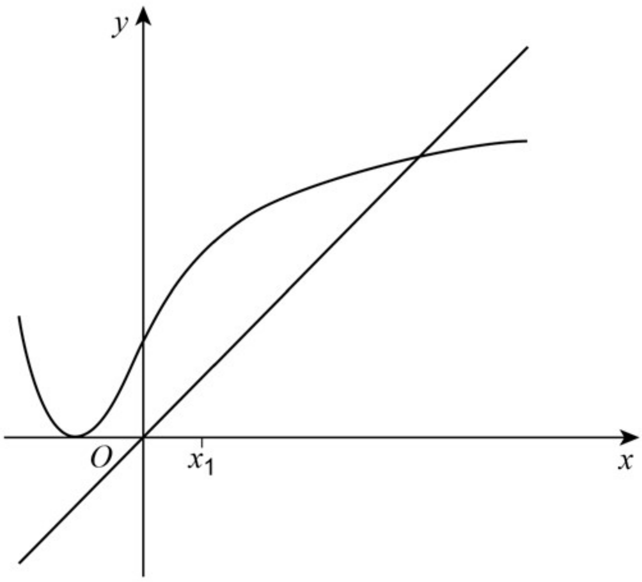
$$x_{n+1} = \left(2 - e^{-x_n}\right)^2$$

with

$$x_1 = 3.5$$

to find x_2 and x_3 giving your answers to three decimal places.

(c) The diagram shows parts of the graphs of $y = \left(2 - e^{-x}\right)^2$ and $y = x$, and a position of x_1



On the diagram, draw a staircase or cobweb diagram to show how convergence takes place, indicating the positions of x_2 and x_3 on the x-axis.

2 The equation $24x^3 + 36x^2 + 18x - 5 = 0$ has one real root, α

(a) Show that α lies in the interval $0.1 < x < 0.2$

(b) Taking $x_1 = 0.2$ as a first approximation to α, use the Newton-Raphson method to find a second approximation, x_2, to α. Give your answer to four decimal places.

3 The equation $x^3 - x^2 - 2 = 0$ has a single root between 1 and 2

Starting with $x_1 = 1$ which of the following iterative formulae will not converge on the root?

Circle your answer.

$$x_{n+1} = \sqrt{\frac{(x_n)^2 + 2}{x_n}} \qquad x_{n+1} = \sqrt[3]{(x_n)^2 + 2} \qquad x_{n+1} = \frac{2}{(x_n)^2} + 1 \qquad x_{n+1} = \frac{(x_n)^3 - 2}{x_n}$$

4 A curve has equation $y = x^3 - 3x + 3$

(a) Show that the curve intersects the x-axis at the point $(\alpha, 0)$ where $-3 < \alpha < -2$

(b) A student attempts to find α using the Newton-Raphson method with $x_1 = -1$

Explain why the student's method fails.

5 Use the iteration

$$x_{n+1} = \frac{\ln(x_n + 3)}{\ln 3}$$

with

$$x_1 = 0.5$$

to find x_3 to two significant figures.

I3	Understand and use numerical integration of functions, including the use of the trapezium rule and estimating the approximate area under a curve and limits that it must lie between.

Only assessed at A-level

Teaching guidance

Students should be able to:

- understand and use the term ‘ordinate’
- use graphical determination to find whether an approximation over- or underestimates the area, depending on the concavity of the curve
- improve an approximation by increasing the number of ordinates or strips used.

This topic links to H4 and can also be usefully explored using interactives.

Examples

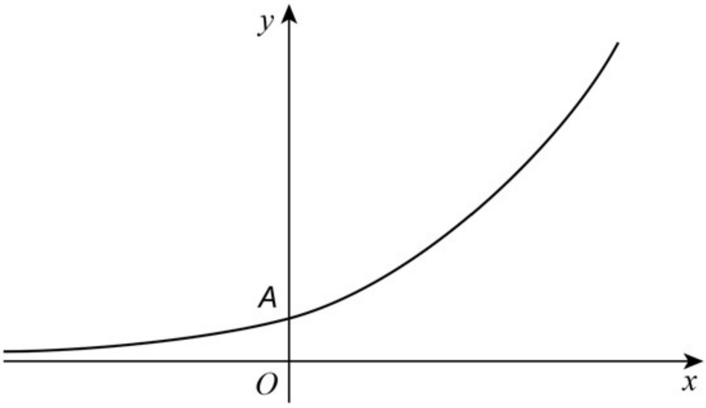
1 (a) Use the trapezium rule with five ordinates to find an approximate value for

$$\int_0^4 \frac{1}{x^2+1}\,\mathrm{d}x$$

giving your answer to four significant figures.

(b) State how you could obtain a better approximation to the value of the integral using the trapezium rule.

2 The diagram shows a sketch of the curve $y = 2^{4x}$



The curve intersects the y-axis at the point *A*.

(a) Find the value of the y-coordinate of *A*.

(b) Use the trapezium rule with six ordinates to find an approximate value for

$$\int_0^1 2^{4x}\,\mathrm{d}x$$

giving your answer to two decimal places.

3 The trapezium rule is used, with six ordinates, to find an estimate for the value of

$$\int_0^5 \mathrm{f}(x)\,\mathrm{d}x$$

Given that f is an increasing function and $\mathrm{f}''(x) > 0$, explain what would happen to the value obtained by the trapezium rule if the number of ordinates was increased.

14 Use numerical methods to solve problems in context.

Only assessed at A-level

Teaching guidance

Students should be able to:

- use a numerical method to solve an equation or to calculate an area resulting from a problem that has its origin within pure mathematics or from a context such as exponential growth or kinematics
- demonstrate that they can apply the correct numerical method in full. Numerical solutions obtained directly from calculator functions are unlikely to achieve full credit.

Examples

1 A particle travels in a straight line with its velocity, v m s^{-1}, at time, t seconds, given by

$$v = 10e^{\frac{t}{5}}\sin t$$

Use the trapezium rule, with 6 ordinates, to estimate the distance moved by the particle in its first 5 seconds of motion.

2 A particle is projected vertically upwards so that its height, h metres, above the ground after time t seconds is given by

$$h = 20t - 9.8t^2 + 2.5e^{-t}$$

(a) How high above the ground was the point where the particle was initially launched?

(b) Given that the particle is in the air for between 1.5 and 2.5 seconds, use the Newton-Raphson method to find the time when the particle hits the ground. Give your answer to two significant figures.

Note: Teachers should remind students of OT2.4: many mathematical problems cannot be solved analytically, but numerical methods permit a solution to a required level of accuracy. This is a subtle and very important point: students are often comfortable with what they consider trial and error methods; we teach them analytical methods that are generally more efficient, but now we are saying that sometimes there are problems that require numerical methods.