MATHS

GCSE TO A LEVEL TRANSITION WORK

Please read through the examples and then try the questions, showing your working clearly.

Indices and Surds

Example 2

Evaluate each of these without using a calculator.

a $25^{0.5}$ **b** 6^{-2} **c** $8^{\frac{2}{3}}$

a $25^{0.5} = 25^{\frac{1}{2}}$
$= \sqrt{25}$ — Since a power of $\frac{1}{2}$ represents a square root.
$= 5$

b $6^{-2} = (6^2)^{-1}$
$= \frac{1}{6^2}$ — Since a power of −1 represents a reciprocal.
$= \frac{1}{36}$

c $8^{\frac{2}{3}} = \left(8^{\frac{1}{3}}\right)^2$ — Always calculate a root before a power.
$= 2^2$ — Since the cube root of 8 is 2
$= 4$

Try It 2

Evaluate each of these without a calculator.

a $36^{\frac{1}{2}}$ **b** $27^{\frac{2}{3}}$ **c** $64^{-0.5}$ **d** $\left(\frac{1}{2}\right)^4$

Example 4

Simplify these expressions without using a calculator.

a $\sqrt{18}+5\sqrt{2}$ **b** $\frac{6}{\sqrt{3}}$ **c** $\frac{2}{1-\sqrt{5}}$

a $\sqrt{18} = \sqrt{9}\sqrt{2}$ — 9 is a square-number factor of 18 so you can simplify $\sqrt{18}$
$= 3\sqrt{2}$
Therefore $\sqrt{18}+5\sqrt{2} = 3\sqrt{2}+5\sqrt{2}$
$= 8\sqrt{2}$ — Collect like terms.

b $\frac{6}{\sqrt{3}} = \frac{6\sqrt{3}}{\sqrt{3}\sqrt{3}}$ — Rationalise the denominator by multiplying numerator and denominator by $\sqrt{3}$
$= \frac{6\sqrt{3}}{3}$
$= 2\sqrt{3}$ — Since $6 \div 3 = 2$

c $\frac{2}{1-\sqrt{5}} = \frac{2(1+\sqrt{5})}{(1-\sqrt{5})(1+\sqrt{5})}$ — Rationalise the denominator by multiplying numerator and denominator by $1+\sqrt{5}$
$= \frac{2(1+\sqrt{5})}{-4}$ — $(1-\sqrt{5})(1+\sqrt{5}) = 1-\sqrt{5}+\sqrt{5}-5$ $= 1-5 = -4$
$= -\frac{1}{2}(1+\sqrt{5})$

Try It 4

Simplify these expressions without using a calculator.

a $3\sqrt{28}-\sqrt{7}$ **b** $\frac{4}{\sqrt{3}}$ **c** $\frac{3}{1+\sqrt{2}}$ **d** $\frac{\sqrt{5}}{\sqrt{5}-2}$

Solving Linear Equations and Rearranging Formulae

Example 3

Rearrange $Ax-3=\frac{x+B}{2}$ to make x the subject.

$2Ax-6=x+B$	Multiply both sides by 2
$2Ax-6-x=B$	Subtract x from both sides.
$2Ax-x=B+6$	Add 6 to both sides.
$x(2A-1)=B+6$	Factorise the side involving x
$x=\frac{B+6}{2A-1}$	Divide both sides by $(2A-1)$ to make x the subject.

Try It 3

Rearrange $3(x+A)=Bx+1$ to make x the subject.

Example 4

Solve the simultaneous equations $5x-4y=17$, $3x+8y=5$

$15x+40y=25$ (1)	Multiply the second equation by 5
$15x-12y=51$ (2)	Multiply the first equation by 3
(1)−(2): $52y=-26$	Subtract equation (2) from equation (1) to eliminate x
$y=-\frac{1}{2}$	
$5x-4\left(-\frac{1}{2}\right)=17$	Substitute $y=-\frac{1}{2}$ into one of the original equations.
$5x+2=17$	Solve this equation to find the value of x
$5x=15$	
$x=3$	

Try It 4

Solve the simultaneous equations $2x+5y=1, 3x-2y=-27$

Example 5

Find the point of intersection between the lines with equations $y=2x+5$ and $y=7-3x$

$2x+5=7-3x$	Substitute $2x+5$ for y in the second equation.
$5x+5=7$	Solve to find the value of x
$5x=2$	
$x=0.4$	
$y=2(0.4)+5$	Substitute $x=0.4$ into either of the original equations to find the y-coordinate.
$=5.8$	
So the lines intersect at the point (0.4, 5.8)	

Try It 5

Find the point of intersection between the lines $y=3x+4$ and $y=6x-2$

Factorising Quadratics and Simple Cubics

Example 1

Factorise each of these quadratics.

a $9x^2+15x$ **b** $x^2+3x-10$ **c** x^2-16

a $9x^2+15x=3x(3x+5)$ — The highest common factor of $9x^2$ and $15x$ is $3x$

b $x^2+3x-10=(x+5)(x-2)$ — You need to find two constants with a product of -10 and a sum of 3: $5\times-2=-10$ and $5+-2=3$ so the constants are -2 and 5

c $x^2-16=(x+4)(x-4)$ — x^2 and 16 are both square numbers.

Try It 1

Factorise each of these quadratics.

a $14x^2-7x$ **b** x^2-5x+4 **c** x^2-25

Example 2

Factorise each of these quadratics.

a $3x^2+11x+6$ **b** $2x^2-9x+10$

a $3x^2+11x+6=3x^2+9x+2x+6$ — Split $11x$ into $9x+2x$ since $9\times2=18$ and $3\times6=18$

$=3x(x+3)+2(x+3)$ — Factorise the first pair of terms and the second pair of terms.

$=(3x+2)(x+3)$

b $2x^2-9x+10=2x^2-4x-5x+10$ — Split $9x$ into $-4x-5x$ since $-4\times-5=20$ and $2\times10=20$

$=2x(x-2)-5(x-2)$ — Factorise the first pair of terms and the second pair of terms.

$=(2x-5)(x-2)$

Try It 2

Factorise each of these quadratics.

a $5x^2+21x+4$ **b** $6x^2+7x-3$ **c** $8x^2-22x+5$

Example 3

Use factorisation to find the roots of these quadratic equations.

a $4x^2+12x=0$ **b** $5x^2=21x-4$

a $4x^2+12x=4x(x+3)$ — Factorise the quadratic.

$4x(x+3)=0\Rightarrow 4x=0$ or $x+3=0$ — One of the factors must be equal to zero.

If $4x=0$ then $x=0$ and if $x+3=0$ then $x=-3$ — Solve to find the roots.

b $5x^2-21x+4=0$ — Rearrange so you have a quadratic expression equal to zero.

$5x^2-21x+4=5x^2-20x-x+4$ — Write $-21x=-x-20x$ since $-20\times-1=20$ and $5\times4=20$

$=5x(x-4)-(x-4)$

$=(5x-1)(x-4)$ — Factorise the quadratic.

$(5x-1)(x-4)=0\Rightarrow 5x-1=0$ or $x-4=0$ — The product is zero so one of the factors must be equal to zero.

If $5x-1=0$ then $x=\frac{1}{5}$ and if $x-4=0$ then $x=4$ — Solve to find the roots.

Try It 3

Find the roots of these quadratic equations.

a $6x^2-12x=0$ **b** $4x^2=23x-15$

Example 4

Sketch these quadratic functions.

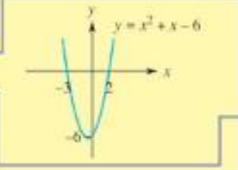
a $y=x^2+x-6$ **b** $y=-x^2+4x$

a When $x=0$, $y=-6$ — Find the y-intercept by letting $x=0$

When $y=0$, $x^2+x-6=0$ — Find the x-intercept by letting $y=0$

$x^2+x-6=(x+3)(x-2)$ — Factorise to find the roots.

$(x+3)(x-2)=0\Rightarrow x=-3$ or $x=2$



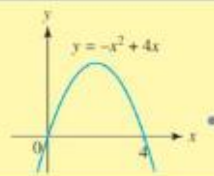
Sketch the parabola and label the y-intercept of -6 and the x-intercepts of -3 and 2

b When $x=0$, $y=0$ — Find the y-intercept by letting $x=0$

When $y=0$, $-x^2+4x=0$ — Find the x-intercept by letting $y=0$

$-x^2+4x=-x(x-4)$ — Factorise to find the roots.

$-x(x-4)=0\Rightarrow x=0$ or $x=4$



Sketch the parabola, it will be this way up since the x^2 term in the quadratic is negative. Label the x and y intercepts.

Try It 4

Sketch these quadratic functions.

a $y=x^2-25$ **b** $y=x^2+10x+25$ **c** $y=5x-x^2$

Completing the Square

Example 1

Write each of these quadratics in the form $p(x+q)^2+r$ where p, q and r are constants to be found.

a x^2+6x+7 **b** $-2x^2+12x$

a $x^2+6x+7=\left(x+\frac{6}{2}\right)^2-\left(\frac{6}{2}\right)^2+7$

$=(x+3)^2-9+7$

$=(x+3)^2-2$

The constant term in the bracket will be half of the coefficient of x

b $-2x^2+12x=-2\left[x^2-6x\right]$

$=-2\left[(x-3)^2-9\right]$

$=-2(x-3)^2+18$

First factor out the coefficient of x^2 then complete the square for the expression in the square brackets.

Try It 1

Write each of these quadratics in the form $p(x+q)^2+r$

a x^2+22x **b** $2x^2-8x-6$ **c** $-x^2+10x$

Example 2

Find the coordinates of the turning point of the curve with equation $y=-x^2+5x-2$

$-x^2+5x-2=-\left[x^2-5x+2\right]$

First factor out the -1 then complete the square for the expression in the square brackets.

$=-\left[\left(x-\frac{5}{2}\right)^2-\frac{25}{4}+2\right]$

$=-\left[\left(x-\frac{5}{2}\right)^2-\frac{17}{4}\right]$

$=-\left(x-\frac{5}{2}\right)^2+\frac{17}{4}$

So the maximum point is at $\left(\frac{5}{2},\frac{17}{4}\right)$

The curve is at its highest point when the bracket is equal to zero: $x-\frac{5}{2}=0 \Rightarrow x=\frac{5}{2}$

Try It 2

Find the coordinates of the turning point of each of these curves and state whether they are a maximum or a minimum.

a $y=x^2-3x+1$ **b** $y=-x^2-7x-12$ **c** $y=2x^2+4x-1$

The Quadratic Formula

Example 1

Solve the equation $3x^2 - 5x - 7 = 0$ using the quadratic formula.

$a = 3, b = -5, c = -7$

Substitute into the formula, taking care with negatives.

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4 \times 3 \times (-7)}}{2 \times 3}$$

$$= \frac{5 \pm \sqrt{109}}{6}$$

$= 2.57$ or -0.91 (to 2 dp)

Use your calculator to give answer as a decimal:

$\frac{5 + \sqrt{109}}{6} = 2.57$ and

$\frac{5 - \sqrt{109}}{6} = -0.91$

You can also use the equation solver on your calculator to solve quadratic equations.

Try It 1

Use the quadratic formula to solve the quadratic equation $7x^2 - 4x - 6 = 0$

Line Graphs

Example 1

Calculate the gradient of the line through the points $A(1, -6)$ and $B(-5, 2)$

$m = \frac{2-(-6)}{(-5)-1}$ — Use $m = \frac{y_2 - y_1}{x_2 - x_1}$ with $x_1 = 1$, $x_2 = -5$ and $y_1 = -6$, $y_2 = 2$

$= \frac{8}{-6}$

$= -\frac{4}{3}$ — The line has a negative gradient so slopes down from left to right.

Try It 1

Find the gradient of the line through each pair of points.

a (1, 7) and (4, 8) **b** (8, –2) and (4, 6) **c** (–8, 7) and (–4, –7)

Example 2

Calculate the exact distance between the point (5, 1) and (6, –4)

$d = \sqrt{(6-5)^2 + (-4-1)^2}$ — Use $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ with $x_1 = 5$, $x_2 = 6$ and $y_1 = 1$, $y_2 = -4$

$= \sqrt{1^2 + (-5)^2}$

$= \sqrt{26}$ — Leave answer as a surd since this is exact.

Try It 2

Calculate the exact distance between each pair of points.

a (5, 2) and (7, 4) **b** (6, –4) and (–3, –1) **c** $(\sqrt{2}, 4)$ and $(4\sqrt{2}, -5)$

Example 3

The points A and B have coordinates (–4, –9) and (6, –2) respectively. Find the midpoint of AB

$\text{Midpoint} = \left(\frac{(-4)+6}{2}, \frac{(-9)+(-2)}{2}\right)$ — Use $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$ with $x_1 = -4$, $x_2 = 6$ and $y_1 = -9$, $y_2 = -2$

$= \left(\frac{2}{2}, \frac{-11}{2}\right)$

$= (1, -5.5)$

Try It 3

Calculate the midpoint of the line segment between each pair of points.

a (1, 9) and (2, 5) **b** (–2, 3) and (–5, –7) **c** (6.4, –9.3) and (–2.6, –3.7)

Example 5

Find the equation of the line through the points (3, 7) and (4, –2) in the form $y = mx + c$

$m = \frac{(-2)-7}{4-3}$ — First use $m = \frac{y_2 - y_1}{x_2 - x_1}$ to find the gradient.

$= -9$

So the equation is $y - 7 = -9(x - 3)$ — Use $y - y_1 = m(x - x_1)$ with $(x_1, y_1) = (3, 7)$, or you could use the point (4, –2) instead.

$y - 7 = -9x + 27$

$y = -9x + 34$ — Expand the brackets and rearrange to the correct form.

Try It 5

Find the equation of the line through each pair of points.

a (3, 7) and (2, 9) **b** (5, –1) and (7, 5) **c** (–3, –4) and (7, 2)

Example 6

The line l_1 has equation $2x+6y=5$. The line l_2 is parallel to l_1 and passes through the point $(1, -5)$. Find the equation of l_2 in the form $ax+by+c=0$ where a, b and c are integers.

$l_1: 2x+6y=5 \Rightarrow 6y=5-2x$

$\Rightarrow y=\frac{5}{6}-\frac{2}{6}x$ — Rearrange to make y the subject so you can see what the gradient is.

The gradient of l_1 is $-\frac{2}{6}$ which simplifies to $-\frac{1}{3}$

Therefore the gradient of l_2 is $-\frac{1}{3}$ — Since l_1 and l_2 are parallel.

So the equation of l_2 is $y-(-5)=-\frac{1}{3}(x-1)$ — Use $y-y_1=m(x-x_1)$ to write the equation of l_2

$\Rightarrow y+5=-\frac{1}{3}(x-1)$

$\Rightarrow -3y-15=x-1$ — Multiply both sides by -3 so that all coefficients are integers.

$\Rightarrow x+3y+14=0$ — Rearrange to the correct form.

Try It 6

The line l_1 has equation $3x-2y=8$. A second line, l_2 is parallel to l_1 and passes through the point $(3, -2)$. Find the equation of l_2 in the form $ax+by+c=0$ where a, b and c are integers.

Example 9

Find the equation of the perpendicular bisector of the line segment joining $(3, -4)$ and $(9, -6)$

Midpoint is $\left(\frac{3+9}{2}, \frac{-4+(-6)}{2}\right)=(6,-5)$ — Use $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

Gradient of line segment is $\frac{-6-(-4)}{9-3}=-\frac{2}{6}=-\frac{1}{3}$ — Use $m=\frac{y_2-y_1}{x_2-x_1}$

So the perpendicular bisector has gradient $m=3$ — Since they are perpendicular and $3\times\left(-\frac{1}{3}\right)=-1$

The equation of the perpendicular bisector is $y-(-5)=3(x-6)$ — Use $y-y_1=m(x-x_1)$

or $y=3x-23$

Try It 9

Find the equation of the perpendicular bisector of the line segment joining $(2, -3)$ and $(-12, 5)$

Circles

Key point

A circle of radius r and centre (a, b) has equation $(x-a)^2+(y-b)^2=r^2$

Example 1

a Find the centre and radius of the circle with equation $(x-5)^2+(y+1)^2=9$

b Write the equation of a circle with centre $(-3, 7)$ and radius 4

a The centre is at $(5, -1)$ — Equation is $(x-5)^2+(y-(-1))^2=9$ so $a=5$ and $b=-1$

The radius is $\sqrt{9}=3$ — Remember to find the positive square root.

b $a=-3$, $b=7$ and $r=4$

So equation is $(x+3)^2+(y-7)^2=16$ — Remember to square the radius.

Try It 1

a Find the centre and radius of the circle with equation $(x+2)^2+(y-8)^2=25$

b Write the equation of a circle with centre $(7, -9)$ and radius 8

Example 2

Find the centre and radius of the circle with equation $x^2+y^2-8x+4y+2=0$

$x^2-8x+y^2+4y+2=0$ — Group the terms involving x and the terms involving y

$(x-4)^2-16+(y+2)^2-4+2=0$ — Complete the square for x^2-8x and y^2+4y

$(x-4)^2+(y+2)^2=18$

So the centre is $(4, -2)$ and the radius is $\sqrt{18}=3\sqrt{2}$

Try It 2

Find the centre and radius of the circles with these equations.

a $x^2+y^2-10y+16=0$ **b** $x^2+y^2+6x-12y=0$

Example 3

Find the equation of the circle with diameter AB where A is $(3, -8)$ and B is $(-5, 4)$

Centre is $\left(\frac{3+(-5)}{2}, \frac{(-8)+4}{2}\right)$
$=(-1, -2)$ — The centre is the midpoint of AB. Use $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

Radius is $\frac{1}{2}\sqrt{(-5-3)^2+(4-(-8))^2}$ — The radius is half of the length of AB
$=\frac{1}{2}\sqrt{(-8)^2+(12)^2}$
$=2\sqrt{13}$

So the equation of the circle is $(x+1)^2+(y+2)^2=52$ — Use $(x-a)^2+(y-b)^2=r^2$ and remember to square the radius: $(2\sqrt{13})^2=52$

Try It 3

Find the equation of the circle with diameter AB where A is $(4, 6)$ and B is $(2, -4)$

Example 4

A circle has equation $(x+3)^2+(y-7)^2=26$

a Show that the point $(-4, 2)$ lies on the circle.

b Find the equation of the tangent to the circle that passes through the point $(-4, 2)$

a $(-4+3)^2+(2-7)^2=(-1)^2+(-5)^2$
$=1+25$ — Substitute $x=-4$, $y=2$ into the equation.
$=26$ so $(-4, 2)$ lies on the circle.

b Centre of circle is $(-3, 7)$

Gradient of radius is $\frac{2-7}{-4-(-3)}=\frac{-5}{-1}=5$ — Use $m=\left(\frac{y_2-y_1}{x_2-x_1}\right)$

A tangent is perpendicular to a radius so gradient of tangent is $-\frac{1}{5}$ — Since $\left(-\frac{1}{5}\right)\times 5=-1$

Therefore equation of tangent is $y-2=-\frac{1}{5}(x+4)$ — Use $y-y_1=m(x-x_1)$ with $(x_1, y_1)=(-4, 2)$

Try It 4

A circle has equation $(x-1)^2+(y+4)^2=50$

a Show that the point $(6, 1)$ lies on the circle.

b Find the equation of the tangent to the circle that passes through the point $(6, 1)$

TRY IT ANSWERS

1. Indices and Surds

a 6 b 9 c $\frac{1}{8}$ d $\frac{1}{16}$

$5\sqrt{7}$ b $\frac{4\sqrt{3}}{3}$ c $-3+3\sqrt{2}$ d $5+2\sqrt{5}$

2. Linear equations and rearranging formulae

$x=\frac{1-3A}{3-B}$

$y=3,\ x=-7$

(2, 10)

3. Factorising Quadratics and simple cubics

1 a $7x(2x-1)$ b $(x-4)(x-1)$ c $(x+5)(x-5)$

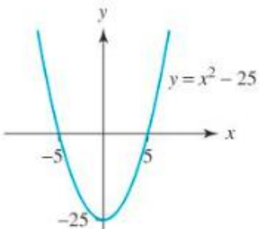
2 a $(5x+1)(x+4)$

b $(3x-1)(2x+3)$

c $(2x-5)(4x-1)$

3 a $x=0$ or $x=2$ b $x=\frac{3}{4}$ or $x=5$

4 a



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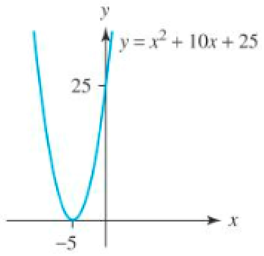

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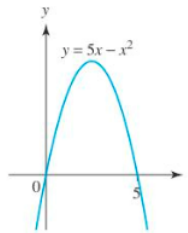

4. Completing The Square

1 a $(x+11)^2-121$ b $2(x-2)^2-14$

c $-(x-5)^2+25$

2 a $\left(\frac{3}{2},-\frac{5}{4}\right)$ is a minimum

b $\left(-\frac{7}{2},\frac{1}{4}\right)$ is a maximum

c $(-1,-3)$ is a minimum

5. The Quadratic Formula

$x=1.25$ or $x=-0.68$

6. Line Graphs

1 a $\frac{1}{3}$ b -2 c $-\frac{7}{2}$

2 a $2\sqrt{2}$ b $3\sqrt{10}$ c $3\sqrt{11}$

3 a (1.5, 7) b (−3.5, −2) c (1.9, −6.5)

5 a $y=-2x+13$ b $y=3x-16$ c $5y=3x-11$

6 $3x-2y-13=0$

9 $7x-4y+39=0$

7. Circles

1 a centre (−2, 8), radius is 5

b $(x-7)^2+(y+9)^2=64$

2 a centre (0, 5), radius 3

b centre (−3, 6), radius $3\sqrt{5}$

3 $(x-3)^2+(y-1)^2=26$

4 a $(6-1)^2+(1+4)^2=5^2+5^2=50$ so (6, 1) lies on the circle

b $y=-x+7$