LONGBENTON HIGH SCHOOL

MATHS

GCSE TO A LEVEL TRANSITION WORK

Please read through the examples and then try the questions, showing your working clearly.

Indices and Surds

Example 1

Simplify these expressions. **a** $2x^3 \times 3x^5$ **b** $12x^7 \div 4x^3$ **c** $(3x^5)^3$

a $2x^3 \times 3x^5 = 6x^{3+5}$

$= 6x^8$

Multiply the coefficients together and use $x^a \times x^b = x^{a+b}$

b $12x^7 \div 4x^6 = \dfrac{12x^7}{4x^6}$

$= 3x$

Since $\frac{12}{4} = 3$ and $x^a \div x^b = x^{a-b}$ so $\frac{x^7}{x^6} = x^1$ which we just write as x

c $(3x^5)^3 = 3^3(x^5)^3$

$= 27x^{15}$

Both the 3 and the x^5 must be raised to the power 3

Since $(x^a)^b = x^{ab}$

Try It 1

Simplify these expressions.

a $5x^3 \times 2x^7$ **b** $18x^9 \div 3x^2$ **c** $(2x^6)^4$ **d** $\left(\dfrac{x^3}{3}\right)^2$

Example 2

Evaluate each of these without using a calculator.

a $25^{0.5}$ **b** 6^{-2} **c** $8^{\frac{2}{3}}$

a $25^{0.5} = 25^{\frac{1}{2}}$

$= \sqrt{25}$

$= 5$

Since a power of $\frac{1}{2}$ represents a square root.

b $6^{-2} = (6^2)^{-1}$

$= \dfrac{1}{6^2}$

$= \dfrac{1}{36}$

Since a power of −1 represents a reciprocal.

c $8^{\frac{2}{3}} = \left(8^{\frac{1}{3}}\right)^2$

$= 2^2$

$= 4$

Always calculate a root before a power.

Since the cube root of 8 is 2

Try It 2

Evaluate each of these without a calculator.

a $36^{\frac{1}{2}}$ **b** $27^{\frac{2}{3}}$ **c** $64^{-0.5}$ **d** $\left(\dfrac{1}{2}\right)^4$

Example 3

Write these expressions in simplified index form.

a $\sqrt[3]{x}$ **b** $\dfrac{2}{x^3}$ **c** $\dfrac{2x}{\sqrt{x}}$

a $\sqrt[3]{x} = x^{\frac{1}{3}}$

b $\dfrac{2}{x^3} = 2x^{-3}$

c $\dfrac{2x}{\sqrt{x}} = \dfrac{2x}{x^{\frac{1}{2}}}$

Since $\sqrt{x} = x^{\frac{1}{2}}$

$= 2x^{1-\frac{1}{2}}$

Subtract the powers, remembering that $x = x^1$

$= 2x^{\frac{1}{2}}$

Try It 3

Write these expressions in simplified index form.

a $\sqrt[5]{x^2}$ **b** $\dfrac{3}{\sqrt{x}}$ **c** $\dfrac{3x^2}{\sqrt{x}}$ **d** $\dfrac{\sqrt{x}}{3x}$

Example 4

Simplify these expressions without using a calculator.

a $\sqrt{18} + 5\sqrt{2}$ **b** $\dfrac{6}{\sqrt{3}}$ **c** $\dfrac{2}{1-\sqrt{5}}$

a $\sqrt{18} = \sqrt{9}\sqrt{2}$

$= 3\sqrt{2}$

9 is a square-number factor of 18 so you can simplify $\sqrt{18}$

Therefore $\sqrt{18} + 5\sqrt{2} = 3\sqrt{2} + 5\sqrt{2}$

$= 8\sqrt{2}$

Collect like terms.

b $\dfrac{6}{\sqrt{3}} = \dfrac{6\sqrt{3}}{\sqrt{3}\sqrt{3}}$

Rationalise the denominator by multiplying numerator and denominator by $\sqrt{3}$

$= \dfrac{6\sqrt{3}}{3}$

$= 2\sqrt{3}$

Since $6 \div 3 = 2$

c $\dfrac{2}{1-\sqrt{5}} = \dfrac{2(1+\sqrt{5})}{(1-\sqrt{5})(1+\sqrt{5})}$

Rationalise the denominator by multiplying numerator and denominator by $1+\sqrt{5}$

$= \dfrac{2(1+\sqrt{5})}{-4}$

$(1-\sqrt{5})(1+\sqrt{5}) = 1-\sqrt{5}+\sqrt{5}-5$
$= 1-5 = -4$

$= -\dfrac{1}{2}(1+\sqrt{5})$

Try It 4

Simplify these expressions without using a calculator.

a $3\sqrt{28} - \sqrt{7}$ **b** $\dfrac{4}{\sqrt{3}}$ **c** $\dfrac{3}{1+\sqrt{2}}$ **d** $\dfrac{\sqrt{5}}{\sqrt{5}-2}$

Solving Linear Equations and Rearranging Formulae

Example 1

Solve the equation $7x-5=3x-2$

$4x-5=-2$ — Subtract $3x$ from both sides of the equation.

$4x=3$ — Add 5 to both sides of the equation.

$x=\frac{3}{4}$ — Divide both sides of the equation by 4

Try It 1

Solve the equation $3x+8=5x-6$

Example 2

Solve the inequality $5(x-2)\leq 2x+1$

$5x-10\leq 2x+1$ — First expand the brackets.

$3x-10\leq 1$ — Subtract $2x$ from both sides.

$3x\leq 11$ — Add 10 to both sides.

$x\leq\frac{11}{3}$ — Divide both sides by 3

Try It 1

Solve the equation $3x+8=5x-6$

Example 3

Rearrange $Ax-3=\frac{x+B}{2}$ to make x the subject.

$2Ax-6=x+B$ — Multiply both sides by 2

$2Ax-6-x=B$ — Subtract x from both sides.

$2Ax-x=B+6$ — Add 6 to both sides.

$x(2A-1)=B+6$ — Factorise the side involving x

$x=\frac{B+6}{2A-1}$ — Divide both sides by $(2A-1)$ to make x the subject.

Try It 3

Rearrange $3(x+A)=Bx+1$ to make x the subject.

Example 4

Solve the simultaneous equations $5x-4y=17$, $3x+8y=5$

$15x+40y=25$ (1) — Multiply the second equation by 5

$15x-12y=51$ (2) — Multiply the first equation by 3

$(1)-(2): 52y=-26$ — Subtract equation (2) from equation (1) to eliminate x

$y=-\frac{1}{2}$

$5x-4\left(-\frac{1}{2}\right)=17$ — Substitute $y=-\frac{1}{2}$ into one of the original equations.

$5x+2=17$ — Solve this equation to find the value of x

$5x=15$

$x=3$

Try It 4

Solve the simultaneous equations $2x+5y=1, 3x-2y=-27$

Example 5

Find the point of intersection between the lines with equations $y=2x+5$ and $y=7-3x$

$2x+5=7-3x$ — Substitute $2x+5$ for y in the second equation.

$5x+5=7$ — Solve to find the value of x

$5x=2$

$x=0.4$

$y=2(0.4)+5$ — Substitute $x=0.4$ into either of the original equations to find the y-coordinate.

$=5.8$

So the lines intersect at the point $(0.4, 5.8)$

Try It 5

Find the point of intersection between the lines $y=3x+4$ and $y=6x-2$

Factorising Quadratics and Simple Cubics

Example 1

Factorise each of these quadratics.

a $9x^2+15x$ **b** $x^2+3x-10$ **c** x^2-16

a $9x^2+15x=3x(3x+5)$ — The highest common factor of $9x^2$ and $15x$ is $3x$

b $x^2+3x-10=(x+5)(x-2)$ — You need to find two constants with a product of -10 and a sum of 3: $5\times-2=-10$ and $5+-2=3$ so the constants are -2 and 5

c $x^2-16=(x+4)(x-4)$ — x^2 and 16 are both square numbers.

Try It 1

Factorise each of these quadratics.

a $14x^2-7x$ **b** x^2-5x+4 **c** x^2-25

Example 2

Factorise each of these quadratics.

a $3x^2+11x+6$ **b** $2x^2-9x+10$

a $3x^2+11x+6=3x^2+9x+2x+6$ — Split $11x$ into $9x+2x$ since $9\times2=18$ and $3\times6=18$

$=3x(x+3)+2(x+3)$ — Factorise the first pair of terms and the second pair of terms.

$=(3x+2)(x+3)$

b $2x^2-9x+10=2x^2-4x-5x+10$ — Split $9x$ into $-4x-5x$ since $-4\times-5=20$ and $2\times10=20$

$=2x(x-2)-5(x-2)$ — Factorise the first pair of terms and the second pair of terms.

$=(2x-5)(x-2)$

Try It 2

Factorise each of these quadratics.

a $5x^2+21x+4$ **b** $6x^2+7x-3$ **c** $8x^2-22x+5$

Example 3

Use factorisation to find the roots of these quadratic equations.

a $4x^2+12x=0$ **b** $5x^2=21x-4$

a $4x^2+12x=4x(x+3)$ — Factorise the quadratic.

$4x(x+3)=0\Rightarrow4x=0$ or $x+3=0$ — One of the factors must be equal to zero.

If $4x=0$ then $x=0$ and if $x+3=0$ then $x=-3$ — Solve to find the roots.

b $5x^2-21x+4=0$ — Rearrange so you have a quadratic expression equal to zero.

$5x^2-21x+4=5x^2-20x-x+4$ — Write $-21x=-x-20x$ since $-20\times-1=20$ and $5\times4=20$

$=5x(x-4)-(x-4)$

$=(5x-1)(x-4)$ — Factorise the quadratic.

$(5x-1)(x-4)=0\Rightarrow5x-1=0$ or $x-4=0$ — The product is zero so one of the factors must be equal to zero.

If $5x-1=0$ then $x=\frac{1}{5}$ and if $x-4=0$ then $x=4$ — Solve to find the roots.

Try It 3

Find the roots of these quadratic equations.

a $6x^2-12x=0$ **b** $4x^2=23x-15$

Example 4

Sketch these quadratic functions.

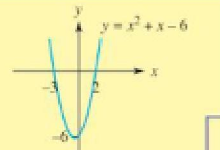
a $y=x^2+x-6$ **b** $y=-x^2+4x$

a When $x=0$, $y=-6$ — Find the y-intercept by letting $x=0$

When $y=0$, $x^2+x-6=0$ — Find the x-intercept by letting $y=0$

$x^2+x-6=(x+3)(x-2)$ — Factorise to find the roots.

$(x+3)(x-2)=0\Rightarrow x=-3$ or $x=2$ — Sketch the parabola and label the y-intercept of -6 and the x-intercepts of -3 and 2

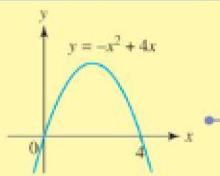


b When $x=0$, $y=0$ — Find the y-intercept by letting $x=0$

When $y=0$, $-x^2+4x=0$ — Find the x-intercept by letting $y=0$

$-x^2+4x=-x(x-4)$ — Factorise to find the roots.

$-x(x-4)=0\Rightarrow x=0$ or $x=4$



Sketch the parabola, it will be this way up since the x^2 term in the quadratic is negative. Label the x and y intercepts.

Try It 4

Sketch these quadratic functions.

a $y=x^2-25$ **b** $y=x^2+10x+25$ **c** $y=5x-x^2$

Completing the Square

Example 1

Write each of these quadratics in the form $p(x+q)^2+r$ where p, q and r are constants to be found.

a x^2+6x+7 **b** $-2x^2+12x$

a $x^2+6x+7=\left(x+\frac{6}{2}\right)^2-\left(\frac{6}{2}\right)^2+7$
$=(x+3)^2-9+7$
$=(x+3)^2-2$

The constant term in the bracket will be half of the coefficient of x

b $-2x^2+12x=-2\left[x^2-6x\right]$
$=-2\left[(x-3)^2-9\right]$
$=-2(x-3)^2+18$

First factor out the coefficient of x^2 then complete the square for the expression in the square brackets.

Try It 1

Write each of these quadratics in the form $p(x+q)^2+r$

a x^2+22x **b** $2x^2-8x-6$ **c** $-x^2+10x$

Example 2

Find the coordinates of the turning point of the curve with equation $y=-x^2+5x-2$

$-x^2+5x-2=-\left[x^2-5x+2\right]$
$=-\left[\left(x-\frac{5}{2}\right)^2-\frac{25}{4}+2\right]$
$=-\left[\left(x-\frac{5}{2}\right)^2-\frac{17}{4}\right]$
$=-\left(x-\frac{5}{2}\right)^2+\frac{17}{4}$

So the maximum point is at $\left(\frac{5}{2},\frac{17}{4}\right)$

First factor out the -1 then complete the square for the expression in the square brackets.

The curve is at its highest point when the bracket is equal to zero: $x-\frac{5}{2}=0 \Rightarrow x=\frac{5}{2}$

Try It 2

Find the coordinates of the turning point of each of these curves and state whether they are a maximum or a minimum.

a $y=x^2-3x+1$ **b** $y=-x^2-7x-12$ **c** $y=2x^2+4x-1$

The Quadratic Formula

Example 1

Solve the equation $3x^2-5x-7=0$ using the quadratic formula.

$a=3, b=-5, c=-7$

$x=\dfrac{-(-5)\pm\sqrt{(-5)^2-4\times3\times(-7)}}{2\times3}$

$=\dfrac{5\pm\sqrt{109}}{6}$

$=2.57$ or -0.91 (to 2 dp)

Substitute into the formula, taking care with negatives.

Use your calculator to give answer as a decimal: $\dfrac{5+\sqrt{109}}{6}=2.57$ and $\dfrac{5-\sqrt{109}}{6}=-0.91$

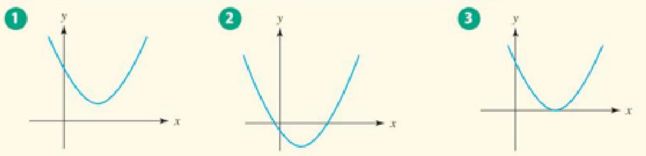
You can also use the equation solver on your calculator to solve quadratic equations.

Try It 1

Use the quadratic formula to solve the quadratic equation $7x^2-4x-6=0$

Key point

1. If $b^2-4ac<0$ then the equation has no real roots.
2. If $b^2-4ac>0$ then the equation has two real roots.
3. If $b^2-4ac=0$ then the equation has one real root.



1: The curve does not cross the x-axis so the discriminant is negative.

2: The curve crosses the x-axis twice so the discriminant is positive.

3: The curve touches the x-axis once so the discriminant equals zero.

Example 2

Given that the quadratic equation $x^2+3x+k+1=0$ has exactly one solution, find the value of k

$a=1, b=3, c=k+1$

So $b^2-4ac=3^2-4\times1\times(k+1)$

$=5-4k$

$5-4k=0 \Rightarrow k=\dfrac{5}{4}$

Find the discriminant.

The equation has exactly one solution so the discriminant is zero.

Try It 2

Given that the quadratic equation $kx^2-x+5=0$ has exactly one solution, find the value of k

Example 3

Given that the quadratic equation $5x^2+3x-k=0$ has real solutions, find the range of possible values of k

$a=5, b=3, c=-k$

So $b^2-4ac=5^2-4\times5\times(-k)$

$=25+20k$

$25+20k\geq0 \Rightarrow k\geq-\dfrac{5}{4}$

Find the discriminant.

The equation has real solutions so the discriminant is greater than or equal to zero.

Try It 2

Given that the quadratic equation $kx^2-x+5=0$ has exactly one solution, find the value of k

Example 4

Given that the quadratic equation $-x^2+7x+3-k=0$ has no real solutions, find the range of possible values of k

$a=-1, b=7, c=3-k$

So $b^2-4ac=7^2-4\times(-1)\times(3-k)$

$=61-4k$

$61-4k<0 \Rightarrow k>\dfrac{61}{4}$

Find the discriminant.

The equation has no solutions so the discriminant is negative.

Try It 4

Given that the quadratic equation $kx^2-7x+1=0$ has no real solutions, find the range of possible values of k

Line Graphs

Example 1

Calculate the gradient of the line through the points $A(1, -6)$ and $B(-5, 2)$

$m = \frac{2-(-6)}{(-5)-1}$

$= \frac{8}{-6}$

$= -\frac{4}{3}$

Use $m = \frac{y_2 - y_1}{x_2 - x_1}$ with $x_1 = 1$, $x_2 = -5$ and $y_1 = -6$, $y_2 = 2$

The line has a negative gradient so slopes down from left to right.

Try It 1

Find the gradient of the line through each pair of points.

a (1, 7) and (4, 8) **b** (8, –2) and (4, 6) **c** (–8, 7) and (–4, –7)

Example 2

Calculate the exact distance between the point (5, 1) and (6, –4)

$d = \sqrt{(6-5)^2 + (-4-1)^2}$

$= \sqrt{1^2 + (-5)^2}$

$= \sqrt{26}$

Use $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ with $x_1 = 5$, $x_2 = 6$ and $y_1 = 1$, $y_2 = -4$

Leave answer as a surd since this is **exact**.

Try It 2

Calculate the exact distance between each pair of points.

a (5, 2) and (7, 4) **b** (6, –4) and (–3, –1) **c** $(\sqrt{2}, 4)$ and $(4\sqrt{2}, -5)$

Example 3

The points A and B have coordinates (–4, –9) and (6, –2) respectively. Find the midpoint of AB

$\text{Midpoint} = \left(\frac{(-4)+6}{2}, \frac{(-9)+(-2)}{2}\right)$

$= \left(\frac{2}{2}, \frac{-11}{2}\right)$

$= (1, -5.5)$

Use $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$ with $x_1 = -4$, $x_2 = 6$ and $y_1 = -9$, $y_2 = -2$

Try It 3

Calculate the midpoint of the line segment between each pair of points.

a (1, 9) and (2, 5) **b** (–2, 3) and (–5, –7) **c** (6.4, –9.3) and (–2.6, –3.7)

Example 4

Work out the gradient and the y-intercept of each of these lines.

a $y = \frac{1}{2}x + 4$ **b** $y + x = 5$ **c** $-2x + 3y + 7 = 0$

a Gradient $= \frac{1}{2}$ and y-intercept $= 4$

Since $y = mx + c$ where m is the gradient and c is the y-intercept.

b $y = 5 - x$

Rearrange to make y the subject.

So gradient $= -1$ and y-intercept $= 5$

c $3y = -7 + 2x$

Rearrange to make y the subject.

$y = -\frac{7}{3} + \frac{2}{3}x$

So gradient $= \frac{2}{3}$ and y-intercept $= -\frac{7}{3}$

Try It 4

Work out the gradient and the y-intercept of each line.

a $y = 8 - 2x$ **b** $2y + x = 3$ **c** $6x - 9y - 4 = 0$

Example 5

Find the equation of the line through the points (3, 7) and (4, –2) in the form $y = mx + c$

$m = \frac{(-2)-7}{4-3}$

$= -9$

First use $m = \frac{y_2 - y_1}{x_2 - x_1}$ to find the gradient.

So the equation is $y - 7 = -9(x - 3)$

Use $y - y_1 = m(x - x_1)$ with $(x_1, y_1) = (3, 7)$, or you could use the point (4, –2) instead.

$y - 7 = -9x + 27$

$y = -9x + 34$

Expand the brackets and rearrange to the correct form.

Try It 5

Find the equation of the line through each pair of points.

a (3, 7) and (2, 9) **b** (5, –1) and (7, 5) **c** (–3, –4) and (7, 2)

Example 6

The line l_1 has equation $2x+6y=5$. The line l_2 is parallel to l_1 and passes through the point $(1, -5)$. Find the equation of l_2 in the form $ax+by+c=0$ where a, b and c are integers.

$l_1: 2x+6y=5 \Rightarrow 6y=5-2x$

$\Rightarrow y=\frac{5}{6}-\frac{2}{6}x$ — Rearrange to make y the subject so you can see what the gradient is.

The gradient of l_1 is $-\frac{2}{6}$ which simplifies to $-\frac{1}{3}$

Therefore the gradient of l_2 is $-\frac{1}{3}$ — Since l_1 and l_2 are parallel.

So the equation of l_2 is $y-(-5)=-\frac{1}{3}(x-1)$ — Use $y-y_1=m(x-x_1)$ to write the equation of l_2

$\Rightarrow y+5=-\frac{1}{3}(x-1)$

$\Rightarrow -3y-15=x-1$ — Multiply both sides by -3 so that all coefficients are integers.

$\Rightarrow x+3y+14=0$ — Rearrange to the correct form.

Try It 6

The line l_1 has equation $3x-2y=8$. A second line, l_2 is parallel to l_1 and passes through the point $(3, -2)$. Find the equation of l_2 in the form $ax+by+c=0$ where a, b and c are integers.

Example 7

Decide whether or not each line is parallel or perpendicular to the line $y=4x-1$

a $2x+8y=5$ **b** $20x+5y=2$ **c** $16x-4y=5$

First note that the gradient of $y=4x-1$ is 4

a $2x+8y=5 \Rightarrow 8y=5-2x$ — Rearrange to make y the subject.

$\Rightarrow y=\frac{5}{8}-\frac{1}{4}x$ — The gradient is $-\frac{1}{4}$

$4\times\left(-\frac{1}{4}\right)=-1$ so this line is perpendicular to $y=4x-1$ — Since the product of the gradients is -1

b $20x+5y=2 \Rightarrow 5y=2-20x$

$\Rightarrow y=\frac{2}{5}-4x$ — Rearrange to make y the subject.

The gradient is -4 so this line is neither parallel nor perpendicular to $y=4x-1$

c $16x-4y=5 \Rightarrow 4y=16x-5$

$\Rightarrow y=4x-\frac{5}{4}$

The gradient is 4 so this line is parallel to $y=4x-1$

Try It 7

Decide whether or not each line is parallel or perpendicular to the line $y=4-3x$

a $3x+6y=2$ **b** $5x-15y=7$ **c** $18x+6y+5=0$

Example 8

The line l_1 has equation $7x+4y=8$ The line l_2 is perpendicular to l_1 and passes through the point $(7, 3)$. Find the equation of l_2 in the form $ax+by+c=0$ where a, b and c are integers.

$l_1: 7x+4y=8 \Rightarrow 4y=-7x+8$

$\Rightarrow y=-\frac{7}{4}x+2$ — Rearrange to make y the subject so you can see what the gradient is.

So the gradient of l_1 is $-\frac{7}{4}$ and the gradient of l_2 is $\frac{4}{7}$ — Since $\left(-\frac{7}{4}\right)\times\frac{4}{7}=-1$

So the equation of l_2 is $y-3=\frac{4}{7}(x-7)$ — Use $y-y_1=m(x-x_1)$ to write the equation of l_2

$\Rightarrow 7y-21=4(x-7)$ — Multiply both sides by 7 so that all coefficients are integers.

$\Rightarrow 7y-21=4x-28$

$\Rightarrow 4x-7y-7=0$ — Rearrange to the correct form.

Try It 8

The line l_1 has equation $4x+6y=3$. A second line, l_2 is perpendicular to l_1 and passes through the point $(-1, 5)$. Find the equation of l_2 in the form $ax+by+c=0$ where a, b and c are integers.

Example 9

Find the equation of the perpendicular bisector of the line segment joining $(3, -4)$ and $(9, -6)$

Midpoint is $\left(\frac{3+9}{2}, \frac{-4+(-6)}{2}\right)=(6, -5)$ — Use $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

Gradient of line segment is $\frac{-6-(-4)}{9-3}=-\frac{2}{6}=-\frac{1}{3}$ — Use $m=\frac{y_2-y_1}{x_2-x_1}$

So the perpendicular bisector has gradient $m=3$ — Since they are perpendicular and $3\times\left(-\frac{1}{3}\right)=-1$

The equation of the perpendicular bisector is $y-(-5)=3(x-6)$ — Use $y-y_1=m(x-x_1)$

or $y=3x-23$

Try It 9

Find the equation of the perpendicular bisector of the line segment joining $(2, -3)$ and $(-12, 5)$

Circles

Key point

A circle of radius r and centre (a, b) has equation $(x-a)^2+(y-b)^2=r^2$

Example 1

a Find the centre and radius of the circle with equation $(x-5)^2+(y+1)^2=9$

b Write the equation of a circle with centre $(-3, 7)$ and radius 4

a The centre is at $(5,-1)$ — Equation is $(x-5)^2+(y-(-1))^2=9$ so $a=5$ and $b=-1$

The radius is $\sqrt{9}=3$ — Remember to find the positive square root.

b $a=-3$, $b=7$ and $r=4$

So equation is $(x+3)^2+(y-7)^2=16$ — Remember to square the radius.

Try It 1

a Find the centre and radius of the circle with equation $(x+2)^2+(y-8)^2=25$

b Write the equation of a circle with centre $(7, -9)$ and radius 8

Example 2

Find the centre and radius of the circle with equation $x^2+y^2-8x+4y+2=0$

$x^2-8x+y^2+4y+2=0$ — Group the terms involving x and the terms involving y

$(x-4)^2-16+(y+2)^2-4+2=0$ — Complete the square for x^2-8x and y^2+4y

$(x-4)^2+(y+2)^2=18$

So the centre is $(4,-2)$ and the radius is $\sqrt{18}=3\sqrt{2}$

Try It 2

Find the centre and radius of the circles with these equations.

a $x^2+y^2-10y+16=0$ **b** $x^2+y^2+6x-12y=0$

Example 3

Find the equation of the circle with diameter AB where A is $(3, -8)$ and B is $(-5, 4)$

Centre is $\left(\frac{3+(-5)}{2}, \frac{(-8)+4}{2}\right)$ $=(-1,-2)$ — The centre is the midpoint of AB. Use $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

Radius is $\frac{1}{2}\sqrt{(-5-3)^2+(4-(-8))^2}$ — The radius is half of the length of AB

$=\frac{1}{2}\sqrt{(-8)^2+(12)^2}$

$=2\sqrt{13}$

So the equation of the circle is $(x+1)^2+(y+2)^2=52$ — Use $(x-a)^2+(y-b)^2=r^2$ and remember to square the radius: $(2\sqrt{13})^2=52$

Try It 3

Find the equation of the circle with diameter AB where A is $(4, 6)$ and B is $(2, -4)$

Example 4

A circle has equation $(x+3)^2+(y-7)^2=26$

a Show that the point $(-4, 2)$ lies on the circle.

b Find the equation of the tangent to the circle that passes through the point $(-4, 2)$

a $(-4+3)^2+(2-7)^2=(-1)^2+(-5)^2$ $=1+25$ $=26$ so $(-4, 2)$ lies on the circle. — Substitute $x=-4$, $y=2$ into the equation.

b Centre of circle is $(-3, 7)$

Gradient of radius is $\frac{2-7}{-4-(-3)}=\frac{-5}{-1}=5$ — Use $m=\left(\frac{y_2-y_1}{x_2-x_1}\right)$

A tangent is perpendicular to a radius so gradient of tangent is $-\frac{1}{5}$ — Since $\left(-\frac{1}{5}\right)\times 5=-1$

Therefore equation of tangent is $y-2=-\frac{1}{5}(x+4)$ — Use $y-y_1=m(x-x_1)$ with $(x_1,y_1)=(-4, 2)$

Try It 4

A circle has equation $(x-1)^2+(y+4)^2=50$

a Show that the point $(6, 1)$ lies on the circle.

b Find the equation of the tangent to the circle that passes through the point $(6, 1)$

Example 5

The line $x+3y=12$ and the circle $(x+3)^2+(y-7)^2=4$ intersect at the points A and B

a Find the coordinates of A and B

b Calculate the length of the chord AB

a $x=12-3y$ — Rearrange the equation of the line to make either x or y the subject (whichever is easiest).

$(12-3y+3)^2+(y-7)^2=4$ — Substitute for x (or y) in the equation of the circle.

$\Rightarrow (15-3y)^2+(y-7)^2=4$

$\Rightarrow 225-90y+9y^2+y^2-14y+49=4$

$\Rightarrow 10y^2-104y+270=0$ — Simplify, then use the equation solver on your calculator.

$\Rightarrow y=5.4$ or $y=5$

$x=12-3(5.4) \Rightarrow x=-4.2$ — Substitute the values of y into the rearranged equation of the line to find the values of x

$x=12-3(5)=-3$

So they intersect at $A(-4.2, 5.4)$ and $B(-3, 5)$ — The line and the circle will intersect twice unless the line is a **tangent** to the circle.

b Length of chord $AB=\sqrt{(-3-(-4.2))^2+(5-5.4)^2}$ — Use $d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

$=\sqrt{1.2^2+(-0.4)^2}$

$=\frac{2}{5}\sqrt{10}$ (= 1.26 to 3 significant figures)

You can find points of intersection using a graphics calculator.

Try It 5

The line $3x+y=5$ intersects the circle $x^2+(y-4)^2=17$ at the points A and B

a Find the coordinates of A and B

b Calculate the length of the chord AB

Example 6

Show that $x-y=12$ is a tangent to the circle $(x-6)^2+(y+2)^2=8$

$y=x-12$ — Rearrange the equation of the line to make either x or y the subject.

$(x-6)^2+(x-12+2)^2=8$ — Substitute for y (or x) into the equation of the circle.

$\Rightarrow (x-6)^2+(x-10)^2=8$

$\Rightarrow x^2-12x+36+x^2-20x+100=8$ — Expand the brackets.

$\Rightarrow 2x^2-32x+128=0$ — Simplify.

$b^2-4ac=(-32)^2-4\times2\times128=0$ — If the discriminant is zero then there is exactly one solution.

So they meet once only.

Hence $x-y=12$ is a tangent to $(x-6)^2+(y+2)^2=8$

To show that a line is a tangent to a circle you can show that they only intersect once.

Try It 6

Show that $2x-y+11=0$ is a tangent to the circle $(x-5)^2+(y-1)^2=80$

TRY IT ANSWERS

1. Indices and Surds

1 a $10x^{10}$ b $6x^7$ c $16x^{24}$ d $\frac{x^6}{9}$

2 a 6 b 9 c $\frac{1}{8}$ d $\frac{1}{16}$

3 a $x^{\frac{2}{5}}$ b $3x^{-\frac{1}{2}}$ c $3x^{\frac{3}{2}}$ d $\frac{1}{3}x^{-\frac{1}{2}}$

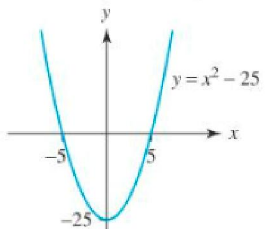
4 a $5\sqrt{7}$ b $\frac{4\sqrt{3}}{3}$ c $-3+3\sqrt{2}$ d $5+2\sqrt{5}$

2. Linear equations and rearranging formulae

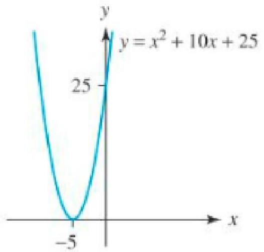
1 $x=7$
2 $x>2$
3 $x=\frac{1-3A}{3-B}$
4 $y=3$, $x=-7$
5 (2, 10)

3. Factorising Quadratics and simple cubics

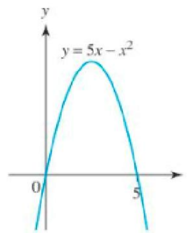
1 a $7x(2x-1)$ b $(x-4)(x-1)$ c $(x+5)(x-5)$
2 a $(5x+1)(x+4)$
b $(3x-1)(2x+3)$
c $(2x-5)(4x-1)$
3 a $x=0$ or $x=2$ b $x=\frac{3}{4}$ or $x=5$
4 a



b



c



4. Completing The Square

1 a $(x+11)^2-121$ b $2(x-2)^2-14$
c $-(x-5)^2+25$

2 a $\left(\frac{3}{2},-\frac{5}{4}\right)$ is a minimum
b $\left(-\frac{7}{2},\frac{1}{4}\right)$ is a maximum
c $(-1,-3)$ is a minimum

5. The Quadratic Formula

1 $x=1.25$ or $x=-0.68$
2 $\frac{1}{20}$
3 $k\geq-\frac{9}{4}$
4 $k>\frac{49}{4}$

6. Line Graphs

1 a $\frac{1}{3}$ b -2 c $-\frac{7}{2}$
2 a $2\sqrt{2}$ b $3\sqrt{10}$ c $3\sqrt{11}$
3 a (1.5, 7) b (–3.5, –2) c (1.9, –6.5)
4 a gradient is –2, y-intercept is 8
b gradient is $-\frac{1}{2}$, y-intercept is $\frac{3}{2}$
c gradient is $\frac{2}{3}$, y-intercept is $-\frac{4}{9}$
5 a $y=-2x+13$ b $y=3x-16$ c $5y=3x-11$
6 $3x-2y-13=0$
7 a neither parallel nor perpendicular
b perpendicular
c parallel
8 $3x-2y+13=0$
9 $7x-4y+39=0$

7. Circles

1 a centre (–2, 8), radius is 5
b $(x-7)^2+(y+9)^2=64$
2 a centre (0, 5), radius 3
b centre (–3, 6), radius $3\sqrt{5}$
3 $(x-3)^2+(y-1)^2=26$
4 a $(6-1)^2+(1+4)^2=5^2+5^2=50$ so (6, 1) lies on the circle
b $y=-x+7$
5 a (1.6, 0.2), (–1, 8)
b $\frac{13}{5}\sqrt{10}$
6 $y=2x+11 \Rightarrow (x-5)^2+(2x+11-1)^2=80$
$\Rightarrow (x-5)^2+(2x+10)^2=80$
$\Rightarrow x^2-10x+25+4x^2+40x+100=80$
$\Rightarrow 5x^2+30x+45=0$

exactly one solution
Therefore the line and the circle touch once, hence the line is a tangent to the circle.