LONGBENTON HIGH SCHOOL

MATHS

GCSE To ALevel

Bridging Booklet

Please read through the examples and then try the questions, showing your working clearly.

Indices and Surds

Example 1

Simplify these expressions. **a** $2x^3 \times 3x^5$ **b** $12x^7 \div 4x^6$ **c** $(3x^5)^3$

a $2x^3 \times 3x^5 = 6x^{3+5}$

$= 6x^8$

Multiply the coefficients together and use $x^a \times x^b = x^{a+b}$

b $12x^7 \div 4x^6 = \dfrac{12x^7}{4x^6}$

$= 3x$

Since $\frac{12}{4} = 3$ and $x^a \div x^b = x^{a-b}$ so $\frac{x^7}{x^6} = x^1$ which we just write as x

c $(3x^5)^3 = 3^3(x^5)^3$

$= 27x^{15}$

Since $(x^a)^b = x^{ab}$

Both the 3 and the x^5 must be raised to the power 3

Try It 1

Simplify these expressions.

a $5x^3 \times 2x^7$ **b** $18x^9 \div 3x^2$ **c** $(2x^6)^4$ **d** $\left(\dfrac{x^3}{3}\right)^2$

Example 2

Evaluate each of these without using a calculator.

a $25^{0.5}$ **b** 6^{-2} **c** $8^{\frac{2}{3}}$

a $25^{0.5} = 25^{\frac{1}{2}}$

$= \sqrt{25}$

$= 5$

Since a power of $\frac{1}{2}$ represents a square root.

b $6^{-2} = (6^2)^{-1}$

$= \dfrac{1}{6^2}$

$= \dfrac{1}{36}$

Since a power of -1 represents a reciprocal.

c $8^{\frac{2}{3}} = \left(8^{\frac{1}{3}}\right)^2$

$= 2^2$

$= 4$

Always calculate a root before a power.

Since the cube root of 8 is 2

Try It 2

Evaluate each of these without a calculator.

a $36^{\frac{1}{2}}$ **b** $27^{\frac{2}{3}}$ **c** $64^{-0.5}$ **d** $\left(\dfrac{1}{2}\right)^4$

Example 3

Write these expressions in simplified index form.

a $\sqrt[3]{x}$ **b** $\dfrac{2}{x^3}$ **c** $\dfrac{2x}{\sqrt{x}}$

a $\sqrt[3]{x} = x^{\frac{1}{3}}$

b $\dfrac{2}{x^3} = 2x^{-3}$

c $\dfrac{2x}{\sqrt{x}} = \dfrac{2x}{x^{\frac{1}{2}}}$

Since $\sqrt{x} = x^{\frac{1}{2}}$

$= 2x^{1-\frac{1}{2}}$

Subtract the powers, remembering that $x = x^1$

$= 2x^{\frac{1}{2}}$

Try It 3

Write these expressions in simplified index form.

a $\sqrt[5]{x^2}$ **b** $\dfrac{3}{\sqrt{x}}$ **c** $\dfrac{3x^2}{\sqrt{x}}$ **d** $\dfrac{\sqrt{x}}{3x}$

Example 4

Simplify these expressions without using a calculator.

a $\sqrt{18} + 5\sqrt{2}$ **b** $\dfrac{6}{\sqrt{3}}$ **c** $\dfrac{2}{1-\sqrt{5}}$

a $\sqrt{18} = \sqrt{9}\sqrt{2}$

9 is a square-number factor of 18 so you can simplify $\sqrt{18}$

$= 3\sqrt{2}$

Therefore $\sqrt{18} + 5\sqrt{2} = 3\sqrt{2} + 5\sqrt{2}$

$= 8\sqrt{2}$

Collect like terms.

b $\dfrac{6}{\sqrt{3}} = \dfrac{6\sqrt{3}}{\sqrt{3}\sqrt{3}}$

Rationalise the denominator by multiplying numerator and denominator by $\sqrt{3}$

$= \dfrac{6\sqrt{3}}{3}$

$= 2\sqrt{3}$

Since $6 \div 3 = 2$

c $\dfrac{2}{1-\sqrt{5}} = \dfrac{2(1+\sqrt{5})}{(1-\sqrt{5})(1+\sqrt{5})}$

Rationalise the denominator by multiplying numerator and denominator by $1+\sqrt{5}$

$= \dfrac{2(1+\sqrt{5})}{-4}$

$(1-\sqrt{5})(1+\sqrt{5}) = 1 - \sqrt{5} + \sqrt{5} - 5$
$= 1 - 5 = -4$

$= -\dfrac{1}{2}(1+\sqrt{5})$

Try It 4

Simplify these expressions without using a calculator.

a $3\sqrt{28} - \sqrt{7}$ **b** $\dfrac{4}{\sqrt{3}}$ **c** $\dfrac{3}{1+\sqrt{2}}$ **d** $\dfrac{\sqrt{5}}{\sqrt{5}-2}$

Exercise

1 Evaluate each of these without using a calculator.

a $49^{\frac{1}{2}}$ b $27^{\frac{1}{3}}$ c 5^{-1} d $64^{-\frac{1}{3}}$

e $9^{\frac{3}{2}}$ f $16^{\frac{3}{4}}$ g $125^{-\frac{2}{3}}$ h $\left(\frac{1}{2}\right)^{3}$

i $\left(\frac{1}{9}\right)^{-2}$ j $\left(\frac{4}{9}\right)^{\frac{1}{2}}$ k $\left(\frac{9}{16}\right)^{-0.5}$ l $\left(\frac{27}{8}\right)^{-\frac{2}{3}}$

2 Simplify these expressions fully without using a calculator.

a $\sqrt{8}$ b $\sqrt{75}$ c $2\sqrt{24}$ d $3\sqrt{48}$

e $\sqrt{20}+\sqrt{5}$ f $\sqrt{27}-\sqrt{12}$ g $5\sqrt{32}-3\sqrt{8}$ h $\sqrt{50}+3\sqrt{125}$

i $\sqrt{68}+3\sqrt{17}$ j $3\sqrt{72}-\sqrt{32}$ k $4\sqrt{18}-2\sqrt{3}$ l $6\sqrt{5}+\sqrt{50}$

3 Simplify these expressions fully without using a calculator.

a $\frac{1}{\sqrt{7}}$ b $\frac{2}{\sqrt{8}}$ c $\frac{12}{\sqrt{3}}$ d $\frac{\sqrt{8}}{\sqrt{12}}$

e $\frac{1}{1+\sqrt{3}}$ f $\frac{2}{1+\sqrt{2}}$ g $\frac{8}{1-\sqrt{5}}$ h $\frac{2}{\sqrt{5}-1}$

i $\frac{\sqrt{2}}{2+\sqrt{3}}$ j $\frac{2\sqrt{3}}{\sqrt{6}-2}$ k $\frac{1+\sqrt{2}}{1-\sqrt{2}}$ l $\frac{3+\sqrt{5}}{\sqrt{5}-3}$

4 Expand the brackets and fully simplify each expression.

a $(1+\sqrt{2})(3+\sqrt{2})$ b $(1+\sqrt{2})(3-\sqrt{2})$ c $(1-\sqrt{2})(3+\sqrt{2})$ d $(1-\sqrt{2})(3-\sqrt{2})$

e $(\sqrt{3}+2)(4+\sqrt{3})$ f $(\sqrt{3}+2)(4-\sqrt{3})$ g $(\sqrt{3}-2)(4+\sqrt{3})$ h $(\sqrt{3}-2)(4-\sqrt{3})$

i $(\sqrt{6}+1)(\sqrt{2}+3)$ j $(\sqrt{6}+1)(\sqrt{2}-3)$ k $(\sqrt{6}-1)(\sqrt{2}+3)$ l $(\sqrt{6}-1)(\sqrt{2}-3)$

5 Write each of these expressions in simplified index form.

a $x^{3}\times x^{2}$ b $7x^{5}\times 3x^{6}$ c $5x^{4}\times 8x^{7}$ d $x^{8}\div x^{2}$

e $8x^{7}\div 2x^{3}$ f $3x^{8}\div 12x^{7}$ g $(x^{5})^{7}$ h $(x^{2})^{-5}$

i $(3x^{2})^{4}$ j $(6x^{3})^{2}$ k $\sqrt{x^{3}}$ l $\sqrt[4]{x^{3}}$

m $\frac{5\sqrt{x}}{x}$ n $2x\sqrt{x}$ o $\frac{x^{2}}{3\sqrt{x}}$ p $x^{3}(x^{3}-1)$

q $x^{3}(\sqrt{x}+2)$ r $\frac{x+2}{x^{3}}$ s $\frac{\sqrt{x}+3}{x}$ t $\frac{(3-x^{3})}{\sqrt{x}}$

u $(\sqrt{x}+3)^{2}$ v $\frac{3+\sqrt{x}}{x^{2}}$ w $\frac{1-x}{2\sqrt{x}}$ x $\frac{\sqrt{x}+2}{3x^{3}}$

Solving Linear Equations and Rearranging Formulae

Example 1

Solve the equation $7x-5=3x-2$

$4x-5=-2$ — Subtract $3x$ from both sides of the equation.

$4x=3$ — Add 5 to both sides of the equation.

$x=\frac{3}{4}$ — Divide both sides of the equation by 4

Try It 1

Solve the equation $3x+8=5x-6$

Example 2

Solve the inequality $5(x-2)\leq 2x+1$

First expand the brackets.

$5x-10\leq 2x+1$ — Subtract $2x$ from both sides.

$3x-10\leq 1$ — Add 10 to both sides.

$3x\leq 11$ — Divide both sides by 3

$x\leq\frac{11}{3}$

Try It 1

Solve the equation $3x+8=5x-6$

Example 3

Rearrange $Ax-3=\frac{x+B}{2}$ to make x the subject.

$2Ax-6=x+B$ — Multiply both sides by 2

$2Ax-6-x=B$ — Subtract x from both sides.

$2Ax-x=B+6$ — Add 6 to both sides.

$x(2A-1)=B+6$ — Factorise the side involving x

$x=\frac{B+6}{2A-1}$ — Divide both sides by $(2A-1)$ to make x the subject.

Try It 3

Rearrange $3(x+A)=Bx+1$ to make x the subject.

Example 4

Solve the simultaneous equations $5x-4y=17$, $3x+8y=5$

$15x+40y=25$ (1) — Multiply the second equation by 5

$15x-12y=51$ (2) — Multiply the first equation by 3

$(1)-(2): 52y=-26$ — Subtract equation (2) from equation (1) to eliminate x

$y=-\frac{1}{2}$

$5x-4\left(-\frac{1}{2}\right)=17$ — Substitute $y=-\frac{1}{2}$ into one of the original equations.

$5x+2=17$ — Solve this equation to find the value of x

$5x=15$

$x=3$

Try It 4

Solve the simultaneous equations $2x+5y=1, 3x-2y=-27$

Example 5

Find the point of intersection between the lines with equations $y=2x+5$ and $y=7-3x$

$2x+5=7-3x$ — Substitute $2x+5$ for y in the second equation.

$5x+5=7$ — Solve to find the value of x

$5x=2$

$x=0.4$

$y=2(0.4)+5$ — Substitute $x=0.4$ into either of the original equations to find the y-coordinate.

$=5.8$

So the lines intersect at the point $(0.4, 5.8)$

Try It 5

Find the point of intersection between the lines $y=3x+4$ and $y=6x-2$

Exercise

1 Solve each of these linear equations.

- **a** $3(2x+9)=7$
- **b** $7-3x=12$
- **c** $\frac{x+4}{5}=7$
- **d** $2x+7=5x-6$
- **e** $8x-3=2(3x+1)$
- **f** $\frac{2x+9}{12}=x-1$
- **g** $2(3x-7)=4x$
- **h** $7-2x=3(4-5x)$

2 Solve each of these linear inequalities.

- **a** $\frac{x}{2}+7\geq 5$
- **b** $3-4x<15$
- **c** $5(x-1)>12+x$
- **d** $\frac{x+1}{3}>2$
- **e** $8x-1\leq 2x-5$
- **f** $3(x+1)\geq\frac{x-3}{2}$
- **g** $3(2x-5)<1-x$
- **h** $x-(3+2x)\geq 2(x+1)$

3 Rearrange each of these formulae to make x the subject.

- **a** $2x+5=3A-1$
- **b** $x+u=vx+3$
- **c** $\frac{3x-1}{k}=2x$
- **d** $5(x-3m)=2nx-4$
- **e** $(1-3x)^2=t$
- **f** $\frac{1}{x}=\frac{1}{p}+\frac{1}{q}$
- **g** $\frac{1}{x^2+k}-6=4$
- **h** $\sqrt{x+A}=2B$

4 Use algebra to solve each of these pairs of simultaneous equations.

- **a** $5x+12y=-6,\ x+5y=4$
- **b** $7x+5y=14,\ 3x+4y=19$
- **c** $2x-5y=4,\ 3x-8y=5$
- **d** $3x-2y=2,\ 8x+3y=4.5$
- **e** $5x-2y=11,\ -2x+3y=22$
- **f** $8x+5y=-0.5,\ -6x+4y=-3.5$

5 Use algebra to find the point of intersection between each pair of lines.

- **a** $y=8-3x,\ y=2-5x$
- **b** $y=7x-4,\ y=3x-2$
- **c** $y=2x+3,\ y=5-x$
- **d** $y+5=3x,\ y=-5x+7$
- **e** $y=\frac{1}{2}x+3,\ y=5-2x$
- **f** $y=3(x+2),\ y=7-2x$

Factorising Quadratics and Simple Cubics

Example 1

Factorise each of these quadratics.

a $9x^2+15x$ **b** $x^2+3x-10$ **c** x^2-16

a $9x^2+15x=3x(3x+5)$

The highest common factor of $9x^2$ and $15x$ is $3x$

b $x^2+3x-10=(x+5)(x-2)$

You need to find two constants with a product of -10 and a sum of 3: $5\times-2=-10$ and $5+-2=3$ so the constants are -2 and 5

c $x^2-16=(x+4)(x-4)$

x^2 and 16 are both square numbers.

Try It 1

Factorise each of these quadratics.

a $14x^2-7x$ **b** x^2-5x+4 **c** x^2-25

Example 2

Factorise each of these quadratics.

a $3x^2+11x+6$ **b** $2x^2-9x+10$

a $3x^2+11x+6=3x^2+9x+2x+6$

Split $11x$ into $9x+2x$ since $9\times2=18$ and $3\times6=18$

$=3x(x+3)+2(x+3)$

Factorise the first pair of terms and the second pair of terms.

$=(3x+2)(x+3)$

b $2x^2-9x+10=2x^2-4x-5x+10$

Split $9x$ into $-4x-5x$ since $-4\times-5=20$ and $2\times10=20$

$=2x(x-2)-5(x-2)$

Factorise the first pair of terms and the second pair of terms.

$=(2x-5)(x-2)$

Try It 2

Factorise each of these quadratics.

a $5x^2+21x+4$ **b** $6x^2+7x-3$ **c** $8x^2-22x+5$

Example 3

Use factorisation to find the roots of these quadratic equations.

a $4x^2+12x=0$ **b** $5x^2=21x-4$

a $4x^2+12x=4x(x+3)$

Factorise the quadratic.

$4x(x+3)=0\Rightarrow 4x=0$ or $x+3=0$

One of the factors must be equal to zero.

If $4x=0$ then $x=0$ and if $x+3=0$ then $x=-3$

Solve to find the roots.

b $5x^2-21x+4=0$

Rearrange so you have a quadratic expression equal to zero.

$5x^2-21x+4=5x^2-20x-x+4$

Write $-21x=-x-20x$ since $-20\times-1=20$ and $5\times4=20$

$=5x(x-4)-(x-4)$

$=(5x-1)(x-4)$

Factorise the quadratic.

$(5x-1)(x-4)=0\Rightarrow 5x-1=0$ or $x-4=0$

The product is zero so one of the factors must be equal to zero.

If $5x-1=0$ then $x=\frac{1}{5}$ and if $x-4=0$ then $x=4$

Solve to find the roots.

Try It 3

Find the roots of these quadratic equations.

a $6x^2-12x=0$ **b** $4x^2=23x-15$

Example 4

Sketch these quadratic functions.

a $y=x^2+x-6$ **b** $y=-x^2+4x$

a When $x=0$, $y=-6$

Find the y-intercept by letting $x=0$

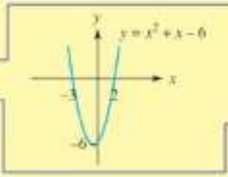
When $y=0$, $x^2+x-6=0$

Find the x-intercept by letting $y=0$

$x^2+x-6=(x+3)(x-2)$

Factorise to find the roots.

$(x+3)(x-2)=0\Rightarrow x=-3$ or $x=2$



Sketch the parabola and label the y-intercept of -6 and the x-intercepts of -3 and 2

b When $x=0$, $y=0$

Find the y-intercept by letting $x=0$

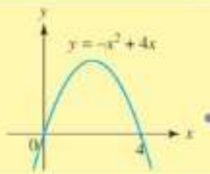
When $y=0$, $-x^2+4x=0$

Find the x-intercept by letting $y=0$

$-x^2+4x=-x(x-4)$

$-x(x-4)=0\Rightarrow x=0$ or $x=4$

Factorise to find the roots.



Sketch the parabola, it will be this way up since the x^2 term in the quadratic is negative. Label the x and y intercepts.

Try It 4

Sketch these quadratic functions.

a $y=x^2-25$ **b** $y=x^2+10x+25$ **c** $y=5x-x^2$

Exercise

1 Fully factorise each of these quadratics.

a $3x^2+5x$ **b** $8x^2-4x$ **c** $17x^2+34x$ **d** $18x^2-24x$

2 Factorise each of these quadratics.

a x^2+5x+6 **b** $x^2-7x+10$ **c** x^2-5x-6 **d** $x^2+3x-28$

e x^2-x-72 **f** $x^2+2x-48$ **g** $x^2-12x+11$ **h** $x^2-5x-24$

3 Factorise each of these quadratics.

a x^2-100 **b** x^2-81 **c** $4x^2-9$ **d** $64-9x^2$

4 Factorise each of these quadratics.

a $3x^2+7x+2$ **b** $6x^2+17x+12$ **c** $4x^2-13x+3$ **d** $2x^2-7x-15$

e $2x^2+3x-5$ **f** $7x^2+25x-12$ **g** $8x^2-22x+15$ **h** $12x^2+17x-5$

5 Fully factorise each of these quadratics.

a $16x^2-25$ **b** $4x^2-16x$ **c** $x^2+13x+12$ **d** $3x^2+16x-35$

e x^2+x-12 **f** $100-9x^2$ **g** $2x^2-14x$ **h** $20x^2-3x-2$

6 Use factorisation to find the roots of these quadratic equations.

a $21x^2-7x=0$ **b** $x^2-36=0$ **c** $17x^2+34x=0$ **d** $6x^2+13x+5=0$

e $4x^2-49=0$ **f** $x^2=7x+18$ **g** $x^2-7x+6=0$ **h** $21x^2=2-x$

i $17x=5x^2+6$ **j** $16x^2+24x+9=0$ **k** $9x^2+4=12x$ **l** $40x^2+x=6$

7 Sketch each of these quadratic functions, labelling where they cross the x and y axes.

a $y=x(x-3)$ **b** $y=-x(3x+2)$ **c** $y=x(3-x)$ **d** $y=(x+2)(x-2)$

e $y=(x+4)^2$ **f** $y=-(2x+5)^2$ **g** $y=(x-5)(x+2)$ **h** $y=(x+1)(5-x)$

8 Sketch each of these quadratic functions, labelling where they cross the x and y axes.

a $y=x^2+6x$ **b** $y=3x^2-12x$ **c** $y=x^2-121$ **d** $y=x^2-3x-10$

e $y=-x^2+3x$ **f** $y=15x-10x^2$ **g** $y=49-x^2$ **h** $y=-x^2+2x+3$

i $y=x^2-4x+4$ **j** $y=-x^2+14x-49$ **k** $y=3x^2+4x+1$ **l** $y=-2x^2+11x-12$

Completing the Square

Example 1

Write each of these quadratics in the form $p(x+q)^2+r$ where p, q and r are constants to be found.

a x^2+6x+7 **b** $-2x^2+12x$

a $x^2+6x+7=\left(x+\frac{6}{2}\right)^2-\left(\frac{6}{2}\right)^2+7$

$=(x+3)^2-9+7$

$=(x+3)^2-2$

The constant term in the bracket will be half of the coefficient of x

b $-2x^2+12x=-2\left[x^2-6x\right]$

$=-2\left[(x-3)^2-9\right]$

$=-2(x-3)^2+18$

First factor out the coefficient of x^2 then complete the square for the expression in the square brackets.

Try It 1

Write each of these quadratics in the form $p(x+q)^2+r$

a x^2+22x **b** $2x^2-8x-6$ **c** $-x^2+10x$

Example 2

Find the coordinates of the turning point of the curve with equation $y=-x^2+5x-2$

$-x^2+5x-2=-\left[x^2-5x+2\right]$

First factor out the -1 then complete the square for the expression in the square brackets.

$=-\left[\left(x-\frac{5}{2}\right)^2-\frac{25}{4}+2\right]$

$=-\left[\left(x-\frac{5}{2}\right)^2-\frac{17}{4}\right]$

$=-\left(x-\frac{5}{2}\right)^2+\frac{17}{4}$

So the maximum point is at $\left(\frac{5}{2},\frac{17}{4}\right)$

The curve is at its highest point when the bracket is equal to zero: $x-\frac{5}{2}=0 \Rightarrow x=\frac{5}{2}$

Try It 2

Find the coordinates of the turning point of each of these curves and state whether they are a maximum or a minimum.

a $y=x^2-3x+1$ **b** $y=-x^2-7x-12$ **c** $y=2x^2+4x-1$

Exercise

1 Write each of these quadratic expressions in the form $p(x+q)^2+r$

a x^2+8x **b** x^2-18x **c** x^2+6x+3 **d** $x^2+12x-5$

e $x^2-7x+10$ **f** x^2+5x+9 **g** $2x^2+8x+4$ **h** $3x^2+18x-6$

i $2x^2-10x+3$ **j** $-x^2+12x-1$ **k** $-x^2+9x-3$ **l** $-2x^2+5x-1$

2 Use completing the square to find the turning point of each of these curves and state whether it is a maximum or a minimum.

a $y=x^2+14x$ **b** $y=x^2-18x+3$ **c** $y=x^2-9x$ **d** $y=-x^2+4x$

e $y=x^2+11x+30$ **f** $y=-x^2+6x-7$ **g** $y=2x^2+16x-5$ **h** $y=-3x^2+15x-2$

The Quadratic Formula

Example 1

Solve the equation $3x^2-5x-7=0$ using the quadratic formula.

$a=3, b=-5, c=-7$

Substitute into the formula, taking care with negatives.

$$x=\frac{-(-5)\pm\sqrt{(-5)^2-4\times3\times(-7)}}{2\times3}$$

$$=\frac{5\pm\sqrt{109}}{6}$$

$=2.57$ or -0.91 (to 2 dp)

Use your calculator to give answer as a decimal: $\frac{5+\sqrt{109}}{6}=2.57$ and $\frac{5-\sqrt{109}}{6}=-0.91$

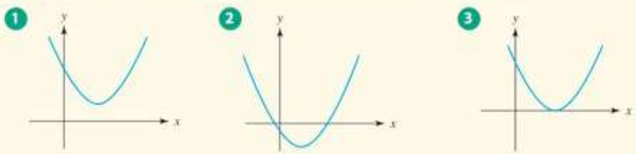
You can also use the equation solver on your calculator to solve quadratic equations.

Try It 1

Use the quadratic formula to solve the quadratic equation $7x^2-4x-6=0$

Key point

1. If $b^2-4ac<0$ then the equation has no real roots.
2. If $b^2-4ac>0$ then the equation has two real roots.
3. If $b^2-4ac=0$ then the equation has one real root.



1. The curve does not cross the x-axis so the discriminant is negative.
2. The curve crosses the x-axis twice so the discriminant is positive.
3. The curve touches the x-axis once so the discriminant equals zero.

Example 2

Given that the quadratic equation $x^2+3x+k+1=0$ has exactly one solution, find the value of k

$a=1, b=3, c=k+1$

So $b^2-4ac=3^2-4\times1\times(k+1)$

Find the discriminant.

$=5-4k$

$5-4k=0\Rightarrow k=\frac{5}{4}$

The equation has exactly one solution so the discriminant is zero.

Try It 2

Given that the quadratic equation $kx^2-x+5=0$ has exactly one solution, find the value of k

Example 3

Given that the quadratic equation $5x^2+3x-k=0$ has real solutions, find the range of possible values of k

$a=5, b=3, c=-k$

So $b^2-4ac=5^2-4\times5\times(-k)$

Find the discriminant.

$=25+20k$

$25+20k\geq0\Rightarrow k\geq-\frac{5}{4}$

The equation has real solutions so the discriminant is greater than or equal to zero.

Try It 2

Given that the quadratic equation $kx^2-x+5=0$ has exactly one solution, find the value of k

Example 4

Given that the quadratic equation $-x^2+7x+3-k=0$ has no real solutions, find the range of possible values of k

$a=-1, b=7, c=3-k$

So $b^2-4ac=7^2-4\times(-1)\times(3-k)$

Find the discriminant.

$=61-4k$

$61-4k<0\Rightarrow k>\frac{61}{4}$

The equation has no solutions so the discriminant is negative.

Try It 4

Given that the quadratic equation $kx^2-7x+1=0$ has no real solutions, find the range of possible values of k

Exercise

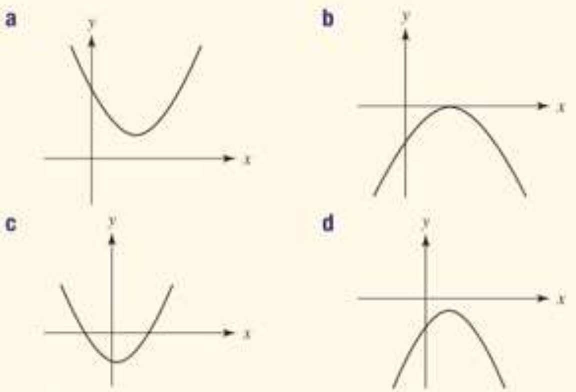
1 Use the quadratic formula to solve each of these equations.

a $7x^2+3x-8=0$ b $-x^2+4x-2=0$ c $x^2-12x+4=0$

2 Work out how many real solutions each of these quadratic equations has.

a $x^2-5x+7=0$ b $7-2x-3x^2=0$ c $4x^2-28x+49=0$

3 Choose a possible equation from the box for each of the graphs.



$y=-4x^2+12x-9$

$y=-x^2+2x-4$

$y=7x^2-5x+4$

$y=-x^2+x+6$

$y=6x^2-x-15$

4 Find the value of k in each equation given that they each have exactly one solution.

a $3x^2+2x-k=0$ b $kx^2-x+4=0$ c $2x^2+5x+k-5=0$

5 Find the range of possible values of k for each equation given that they all have real solutions.

a $x^2+3x-3k=0$ b $kx^2-7x+4=0$ c $-x^2+6x-k-2=0$

6 Find the range of possible values of k for each equation given that they all have no real solutio

a $5x^2-x+2k=0$ b $-kx^2+4x+5=0$ c $6x^2-5x+3-2k=0$

Line Graphs

Example 1

Calculate the gradient of the line through the points $A(1, -6)$ and $B(-5, 2)$

$m = \frac{2-(-6)}{(-5)-1}$

$= \frac{8}{-6}$

$= -\frac{4}{3}$

Use $m = \frac{y_2 - y_1}{x_2 - x_1}$ with $x_1 = 1$, $x_2 = -5$ and $y_1 = -6$, $y_2 = 2$

The line has a negative gradient so slopes down from left to right.

Try It 1

Find the gradient of the line through each pair of points.

a (1, 7) and (4, 8) **b** (8, –2) and (4, 6) **c** (–8, 7) and (–4, –7)

Example 2

Calculate the exact distance between the point (5, 1) and (6, –4)

$d = \sqrt{(6-5)^2 + (-4-1)^2}$

$= \sqrt{1^2 + (-5)^2}$

$= \sqrt{26}$

Use $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ with $x_1 = 5$, $x_2 = 6$ and $y_1 = 1$, $y_2 = -4$

Leave answer as a surd since this is exact.

Try It 2

Calculate the exact distance between each pair of points.

a (5, 2) and (7, 4) **b** (6, –4) and (–3, –1) **c** $(\sqrt{2}, 4)$ and $(4\sqrt{2}, -5)$

Example 3

The points A and B have coordinates (–4, –9) and (6, –2) respectively. Find the midpoint of AB

$\text{Midpoint} = \left(\frac{(-4)+6}{2}, \frac{(-9)+(-2)}{2}\right)$

$= \left(\frac{2}{2}, \frac{-11}{2}\right)$

$= (1, -5.5)$

Use $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$ with $x_1 = -4$, $x_2 = 6$ and $y_1 = -9$, $y_2 = -2$

Try It 3

Calculate the midpoint of the line segment between each pair of points.

a (1, 9) and (2, 5) **b** (–2, 3) and (–5, –7) **c** (6.4, –9.3) and (–2.6, –3.7)

Example 4

Work out the gradient and the y-intercept of each of these lines.

a $y = \frac{1}{2}x + 4$ **b** $y + x = 5$ **c** $-2x + 3y + 7 = 0$

a Gradient $= \frac{1}{2}$ and y-intercept $= 4$

Since $y = mx + c$ where m is the gradient and c is the y-intercept.

b $y = 5 - x$

Rearrange to make y the subject.

So gradient $= -1$ and y-intercept $= 5$

c $3y = -7 + 2x$

Rearrange to make y the subject.

$y = -\frac{7}{3} + \frac{2}{3}x$

So gradient $= \frac{2}{3}$ and y-intercept $= -\frac{7}{3}$

Try It 4

Work out the gradient and the y-intercept of each line.

a $y = 8 - 2x$ **b** $2y + x = 3$ **c** $6x - 9y - 4 = 0$

Example 5

Find the equation of the line through the points (3, 7) and (4, –2) in the form $y = mx + c$

$m = \frac{(-2)-7}{4-3}$

$= -9$

First use $m = \frac{y_2 - y_1}{x_2 - x_1}$ to find the gradient.

So the equation is $y - 7 = -9(x - 3)$

Use $y - y_1 = m(x - x_1)$ with $(x_1, y_1) = (3, 7)$, or you could use the point (4, –2) instead.

$y - 7 = -9x + 27$

$y = -9x + 34$

Expand the brackets and rearrange to the correct form.

Try It 5

Find the equation of the line through each pair of points.

a (3, 7) and (2, 9) **b** (5, –1) and (7, 5) **c** (–3, –4) and (7, 2)

Example 6

The line l_1 has equation $2x+6y=5$. The line l_2 is parallel to l_1 and passes through the point $(1, -5)$. Find the equation of l_2 in the form $ax+by+c=0$ where a, b and c are integers.

$l_1: 2x+6y=5 \Rightarrow 6y=5-2x$

$\Rightarrow y=\frac{5}{6}-\frac{2}{6}x$ — Rearrange to make y the subject so you can see what the gradient is.

The gradient of l_1 is $-\frac{2}{6}$ which simplifies to $-\frac{1}{3}$

Therefore the gradient of l_2 is $-\frac{1}{3}$ — Since l_1 and l_2 are parallel.

So the equation of l_2 is $y-(-5)=-\frac{1}{3}(x-1)$ — Use $y-y_1=m(x-x_1)$ to write the equation of l_2

$\Rightarrow y+5=-\frac{1}{3}(x-1)$

$\Rightarrow -3y-15=x-1$ — Multiply both sides by -3 so that all coefficients are integers.

$\Rightarrow x+3y+14=0$ — Rearrange to the correct form.

Try It 6

The line l_1 has equation $3x-2y=8$. A second line, l_2 is parallel to l_1 and passes through the point $(3, -2)$. Find the equation of l_2 in the form $ax+by+c=0$ where a, b and c are integers.

Example 7

Decide whether or not each line is parallel or perpendicular to the line $y=4x-1$

a $2x+8y=5$ **b** $20x+5y=2$ **c** $16x-4y=5$

First note that the gradient of $y=4x-1$ is 4

a $2x+8y=5 \Rightarrow 8y=5-2x$ — Rearrange to make y the subject.

$\Rightarrow y=\frac{5}{8}-\frac{1}{4}x$ — The gradient is $-\frac{1}{4}$

$4\times\left(-\frac{1}{4}\right)=-1$ so this line is perpendicular to $y=4x-1$ — Since the product of the gradients is -1

b $20x+5y=2 \Rightarrow 5y=2-20x$

$\Rightarrow y=\frac{2}{5}-4x$ — Rearrange to make y the subject.

The gradient is -4 so this line is neither parallel nor perpendicular to $y=4x-1$

c $16x-4y=5 \Rightarrow 4y=16x-5$

$\Rightarrow y=4x-\frac{5}{4}$

The gradient is 4 so this line is parallel to $y=4x-1$

Try It 7

Decide whether or not each line is parallel or perpendicular to the line $y=4-3x$

a $3x+6y=2$ **b** $5x-15y=7$ **c** $18x+6y+5=0$

Example 8

The line l_1 has equation $7x+4y=8$ The line l_2 is perpendicular to l_1 and passes through the point $(7, 3)$. Find the equation of l_2 in the form $ax+by+c=0$ where a, b and c are integers.

$l_1: 7x+4y=8 \Rightarrow 4y=-7x+8$

$\Rightarrow y=-\frac{7}{4}x+2$ — Rearrange to make y the subject so you can see what the gradient is.

So the gradient of l_1 is $-\frac{7}{4}$ and the gradient of l_2 is $\frac{4}{7}$ — Since $\left(-\frac{7}{4}\right)\times\frac{4}{7}=-1$

So the equation of l_2 is $y-3=\frac{4}{7}(x-7)$ — Use $y-y_1=m(x-x_1)$ to write the equation of l_2

$\Rightarrow 7y-21=4(x-7)$ — Multiply both sides by 7 so that all coefficients are integers.

$\Rightarrow 7y-21=4x-28$

$\Rightarrow 4x-7y-7=0$ — Rearrange to the correct form.

Try It 8

The line l_1 has equation $4x+6y=3$. A second line, l_2 is perpendicular to l_1 and passes through the point $(-1, 5)$. Find the equation of l_2 in the form $ax+by+c=0$ where a, b and c are integers.

Example 9

Find the equation of the perpendicular bisector of the line segment joining $(3, -4)$ and $(9, -6)$

Midpoint is $\left(\frac{3+9}{2}, \frac{-4+(-6)}{2}\right)=(6, -5)$ — Use $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

Gradient of line segment is $\frac{-6-(-4)}{9-3}=-\frac{2}{6}=-\frac{1}{3}$ — Use $m=\frac{y_2-y_1}{x_2-x_1}$

So the perpendicular bisector has gradient $m=3$ — Since they are perpendicular and $3\times\left(-\frac{1}{3}\right)=-1$

The equation of the perpendicular bisector is $y-(-5)=3(x-6)$ — Use $y-y_1=m(x-x_1)$

or $y=3x-23$

Try It 9

Find the equation of the perpendicular bisector of the line segment joining $(2, -3)$ and $(-12, 5)$

Exercise

1 Find the gradient of the line through each pair of points.

a $(3, 7)$ and $(2, 8)$ b $(5, 2)$ and $(-4, -6)$ c $(1.3, 4.7)$ and $(2.6, -3.1)$

d $\left(\frac{1}{2}, \frac{1}{3}\right)$ and $\left(\frac{3}{4}, \frac{2}{3}\right)$ e $(\sqrt{3}, 2)$ and $(2\sqrt{3}, 5)$ f $(3a, a)$ and $(a, 5a)$

2 Calculate the exact distance between each pair of points.

a $(8, 4)$ and $(1, 3)$ b $(-3, 9)$ and $(12, -7)$ c $(5.9, 6.2)$ and $(-8.1, 3.8)$

d $\left(\frac{1}{5}, -\frac{1}{5}\right)$ and $\left(\frac{3}{5}, -\frac{4}{5}\right)$ e $(5, -3\sqrt{2})$ and $(2, \sqrt{2})$ f $(k, -3k)$ and $(2k, -6k)$

3 Find the coordinates of the midpoint of each pair of points.

a $(3, 9)$ and $(1, 7)$ b $(2, -4)$ and $(-3, -9)$ c $(2.1, 3.5)$ and $(6.3, -3.7)$

d $\left(\frac{2}{3}, -\frac{1}{2}\right)$ and $\left(-\frac{5}{3}, -\frac{3}{2}\right)$ e $(6\sqrt{5}, 2\sqrt{5})$ and $(-\sqrt{5}, \sqrt{5})$ f $(m, 2n)$ and $(3m, -2n)$

4 Work out the gradient and the y-intercept of these lines.

a $y=7x-4$ b $y+2x=3$ c $x-y=4$ d $3x+2y=7$

e $5x-2y=9$ f $5y-3x=0$ g $x+6y+3=0$ h $3(y-2)=4(x-1)$

5 Find the equation of the line through each pair of points.

a $(2, 5)$ and $(0, 6)$ b $(1, -3)$ and $(2, -5)$ c $(4, 4)$ and $(7, -7)$

d $(8, -2)$ and $(4, -3)$ e $(-3, -7)$ and $(5, 9)$ f $(\sqrt{2}, -\sqrt{2})$ and $(3\sqrt{2}, 4\sqrt{2})$

6 Which of these lines is either parallel or perpendicular to the line with equation $y=6x+5$?

a $2x+12y+3=0$ b $18x+3y=2$ c $3x-\frac{1}{2}y+5=0$

7 Which of these lines is either parallel or perpendicular to the line with equation $y=\frac{2}{3}x-4$?

a $24x+16y+3=0$ b $6x+9y+2=0$ c $2x-3y=7$

8 Which of these lines is either parallel or perpendicular to the line with equation $6x+12y=1$?

a $2y=5-x$ b $9x=18y+4$ c $10x-5y+3=0$

9 The line l_1 has equation $y=5x+1$

a Find the equation of the line l_2 which is parallel to l_1 and passes through $(3, -3)$

b Find the equation of the line l_3 which is perpendicular to l_1 and passes through $(-4, 1)$

14 Find the equation of the perpendicular bisector of the line segment joining each pair of points.

a $(5, -7)$ and $(-3, 5)$ b $(-5, -9)$ and $(5, 5)$ c $(-6, 2)$ and $(4, 12)$

d $(2, -7)$ and $(-1, 2)$ e $(-13, -5)$ and $(15, -12)$

15 Find the point of intersection between these pairs of lines.

a $y=5x-4$ and $y=3-2x$ b $y=8x$ and $y=3x-10$

c $y=7x-5$ and $y=-\frac{1}{2}x+5$ d $y=\frac{1}{4}x+7$ and $y=5x-\frac{5}{2}$

Circles

Key point

A circle of radius r and centre (a, b) has equation $(x-a)^2+(y-b)^2=r^2$

Example 1

a Find the centre and radius of the circle with equation $(x-5)^2+(y+1)^2=9$

b Write the equation of a circle with centre $(-3, 7)$ and radius 4

a The centre is at $(5, -1)$ — Equation is $(x-5)^2+(y-(-1))^2=9$ so $a=5$ and $b=-1$

The radius is $\sqrt{9}=3$ — Remember to find the positive square root.

b $a=-3$, $b=7$ and $r=4$

So equation is $(x+3)^2+(y-7)^2=16$ — Remember to square the radius.

Try It 1

a Find the centre and radius of the circle with equation $(x+2)^2+(y-8)^2=25$

b Write the equation of a circle with centre $(7, -9)$ and radius 8

Example 2

Find the centre and radius of the circle with equation $x^2+y^2-8x+4y+2=0$

$x^2-8x+y^2+4y+2=0$ — Group the terms involving x and the terms involving y

$(x-4)^2-16+(y+2)^2-4+2=0$ — Complete the square for x^2-8x and y^2+4y

$(x-4)^2+(y+2)^2=18$

So the centre is $(4, -2)$ and the radius is $\sqrt{18}=3\sqrt{2}$

Try It 2

Find the centre and radius of the circles with these equations.

a $x^2+y^2-10y+16=0$ **b** $x^2+y^2+6x-12y=0$

Example 3

Find the equation of the circle with diameter AB where A is $(3, -8)$ and B is $(-5, 4)$

Centre is $\left(\frac{3+(-5)}{2}, \frac{(-8)+4}{2}\right)$ — The centre is the midpoint of AB. Use $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

$=(-1, -2)$

Radius is $\frac{1}{2}\sqrt{(-5-3)^2+(4-(-8))^2}$ — The radius is half of the length of AB

$=\frac{1}{2}\sqrt{(-8)^2+(12)^2}$

$=2\sqrt{13}$

So the equation of the circle is $(x+1)^2+(y+2)^2=52$ — Use $(x-a)^2+(y-b)^2=r^2$ and remember to square the radius: $(2\sqrt{13})^2=52$

Try It 3

Find the equation of the circle with diameter AB where A is $(4, 6)$ and B is $(2, -4)$

Example 4

A circle has equation $(x+3)^2+(y-7)^2=26$

a Show that the point $(-4, 2)$ lies on the circle.

b Find the equation of the tangent to the circle that passes through the point $(-4, 2)$

a $(-4+3)^2+(2-7)^2=(-1)^2+(-5)^2$ — Substitute $x=-4$, $y=2$ into the equation.

$=1+25$

$=26$ so $(-4, 2)$ lies on the circle.

b Centre of circle is $(-3, 7)$

Gradient of radius is $\frac{2-7}{-4-(-3)}=\frac{-5}{-1}=5$ — Use $m=\left(\frac{y_2-y_1}{x_2-x_1}\right)$

A tangent is perpendicular to a radius so gradient of tangent is $-\frac{1}{5}$ — Since $\left(-\frac{1}{5}\right)\times 5=-1$

Therefore equation of tangent is $y-2=-\frac{1}{5}(x+4)$ — Use $y-y_1=m(x-x_1)$ with $(x_1, y_1)=(-4, 2)$

Try It 4

A circle has equation $(x-1)^2+(y+4)^2=50$

a Show that the point $(6, 1)$ lies on the circle.

b Find the equation of the tangent to the circle that passes through the point $(6, 1)$

Example 5

The line $x+3y=12$ and the circle $(x+3)^2+(y-7)^2=4$ intersect at the points A and B

a Find the coordinates of A and B

b Calculate the length of the chord AB

a $x=12-3y$ — Rearrange the equation of the line to make either x or y the subject (whichever is easiest).

$(12-3y+3)^2+(y-7)^2=4$ — Substitute for x (or y) in the equation of the circle.

$\Rightarrow (15-3y)^2+(y-7)^2=4$

$\Rightarrow 225-90y+9y^2+y^2-14y+49=4$

$\Rightarrow 10y^2-104y+270=0$ — Simplify, then use the equation solver on your calculator.

$\Rightarrow y=5.4$ or $y=5$

$x=12-3(5.4) \Rightarrow x=-4.2$ — Substitute the values of y into the rearranged equation of the line to find the values of x

$x=12-3(5)=-3$

So they intersect at $A(-4.2, 5.4)$ and $B(-3, 5)$ — The line and the circle will intersect twice unless the line is a **tangent** to the circle.

b Length of chord $AB=\sqrt{(-3-(-4.2))^2+(5-5.4)^2}$ — Use $d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

$=\sqrt{1.2^2+(-0.4)^2}$

$=\frac{2}{5}\sqrt{10}$ (= 1.26 to 3 significant figures)

You can find points of intersection using a graphics calculator.

Try It 5

The line $3x+y=5$ intersects the circle $x^2+(y-4)^2=17$ at the points A and B

a Find the coordinates of A and B

b Calculate the length of the chord AB

Example 6

Show that $x-y=12$ is a tangent to the circle $(x-6)^2+(y+2)^2=8$

$y=x-12$ — Rearrange the equation of the line to make either x or y the subject.

$(x-6)^2+(x-12+2)^2=8$ — Substitute for y (or x) into the equation of the circle.

$\Rightarrow (x-6)^2+(x-10)^2=8$

$\Rightarrow x^2-12x+36+x^2-20x+100=8$ — Expand the brackets.

$\Rightarrow 2x^2-32x+128=0$ — Simplify.

$b^2-4ac=(-32)^2-4\times2\times128=0$ — If the discriminant is zero then there is exactly one solution.

So they meet once only.

Hence $x-y=12$ is a tangent to $(x-6)^2+(y+2)^2=8$

To show that a line is a tangent to a circle you can show that they only intersect once.

Try It 6

Show that $2x-y+11=0$ is a tangent to the circle $(x-5)^2+(y-1)^2=80$

Exercise

1 Write the equations of these circles.

- **a** circle with radius 7 and centre (2, 5)
- **b** circle with radius 4 and centre (–1, –3)
- **c** circle with radius $\sqrt{2}$ and centre (–3, 0)
- **d** circle with radius $\sqrt{5}$ and centre (4, –2)

2 Find the centre and the radius of the circles with these equations.

- **a** $(x-5)^2+(y-3)^2=16$
- **b** $(x+3)^2+(y-4)^2=36$
- **c** $(x-9)^2+(y+2)^2=100$
- **d** $(x+3)^2+(y+1)^2=80$
- **e** $(x-\sqrt{2})^2+(y+2\sqrt{2})^2=32$
- **f** $\left(x+\frac{1}{4}\right)^2+\left(y+\frac{1}{3}\right)^2=\frac{25}{4}$

3 Find the centre and the radius of the circles with these equations.

- **a** $x^2+2x+y^2=24$
- **b** $x^2+y^2+12y=13$
- **c** $x^2+y^2-4x+3=0$
- **d** $x^2+y^2+6x+8y+2=0$
- **e** $x^2+y^2-8x-10y=3$
- **f** $x^2+y^2+14x-2y=5$
- **g** $x^2+y^2+5x-4y+3=0$
- **h** $x^2+y^2-3x-9y=2$
- **i** $x^2+y^2-x+7y+12=0$

4 Find the equation of the circle with diameter AB where the coordinates of A and B are

- **a** (3, 5) and (1, 7)
- **b** (4, –1) and (2, –5)
- **c** (1, –3) and (–9, –6)
- **d** (–3, –7) and (8, –16)
- **e** $(\sqrt{2}, 4)$ and $(-\sqrt{2}, 6)$
- **f** $(4\sqrt{3}, -\sqrt{3})$ and $(-2\sqrt{3}, -5\sqrt{3})$

5 Determine whether each of these points lies on the circle with equation $(x-3)^2+(y+2)^2=5$

- **a** (5, 3)
- **b** (1, –1)
- **c** (4, 3)
- **d** (2, 0)

6 Determine which of these circles the point $(-3, 2)$ lies on.

a $(x-5)^2+y^2=68$ **b** $(x+2)^2+(y+1)^2=8$ **c** $(x-6)^2+(y-2)^2=81$

7 A circle has equation $(x-1)^2+(y+1)^2=10$. Find the equation of the tangent to the circle through the point $(2, -4)$. Write your answer in the form $ax+by+c=0$ where a, b and c are integers.

8 A circle has equation $(x+3)^2+(y+7)^2=34$. Find the equation of the tangent to the circle through the point $(0, -2)$. Write your answer in the form $ax+by+c=0$ where a, b and c are integers.

9 A circle has equation $x^2+(y-8)^2=153$. Find the equation of the tangent to the circle through the point $(3, -4)$. Write your answer in the form $y=mx+c$

11 Find the points of intersection, A and B, between these pairs of lines and circles.

a $x+y=5,\ x^2+y^2=53$ **b** $y+1=0,\ (x-1)^2+(y+2)^2=17$

c $2x-y+7=0,\ (x-2)^2+(y+1)^2=36$ **d** $y=2x+1,\ (x+4)^2+(y+6)^2=10$

12 The line $3x-9y=6$ intersects the circle $(x+7)^2+(y+3)^2=10$ at the points A and B

a Find the coordinates of A and B **b** Calculate the length of the chord AB

14 Show that the line $y=x-3$ is a tangent to the circle $(x-3)^2+(y+2)^2=2$

17 Show that the line $y=2x+3$ does not intersect the circle $(x-1)^2+(y+4)^2=1$

18 Show that the line $3x+4y+2=0$ does not intersect the circle $(x+3)^2+(y-6)^2=9$